Occupation Time Fluctuations in Branching Systems[1]

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Abstract

We consider particle systems in locally compact Abelian groups with particles moving according to a process with symmetric stationary independent increments and undergoing one and two levels of critical branching. We obtain long time fluctuation limits for the occupation time process of the one– and two–level systems. We give complete results for the case of finite variance branching, where the fluctuation limits are Gaussian random fields, and partial results for an example of infinite variance branching, where the fluctuation limits are stable random fields. The asymptotics of the occupation time fluctuations are determined by the Green potential operator G of the individual particle motion and its powers G^2, G^3, and by the growth as $t \to \infty$ of the operator $G_t = \int_0^t T_s ds$ and its powers, where T_t is the semigroup of the motion. The results are illustrated with two examples of motions: the symmetric α–stable Lévy process in $\mathbb{R}^d$ $(0 < \alpha \leq 2)$, and the so called c–hierarchical random walk in the hierarchical group of order N $(0 < c < N)$. We show that the two motions have analogous asymptotics of G_t and its powers that depend on an order parameter γ for their transience/recurrence behavior. This parameter is $\gamma = d/\alpha - 1$ for the α–stable motion, and $\gamma = \log c/\log(N/c)$ for the c–hierarchical random walk. As a consequence of these analogies, the asymptotics of the occupation time fluctuations of the corresponding branching particle systems are also analogous. In the case of the c–hierarchical random walk, however, the growth of G_t and its powers is modulated by oscillations on a logarithmic time scale.

Key words: multilevel branching particle system, occupation time, fluctuation, Green potential, weak and strong transience, stable Lévy process, hierarchical random walk, critical dimensions.

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1. INTRODUCTION

Consider a particle system described by a random counting measure X_t on a space of sites S, with the same intensity measure EX_t for all t which is denoted by ρ, and such that X_t converges in distribution as $t \to \infty$ towards an equilibrium state which also has the same intensity ρ, i.e., the system is *persistent*. Then under mild conditions the *occupation time fluctuation* $Y_t = \int_0^t (X_s - \rho) ds$ obeys a law of large numbers, i.e. $\frac{1}{t} Y_t \to 0$ as $t \to \infty$ (see e.g. Méléard and Roelly[32] for a branching particle system in $\mathbb{R}^d$). The question for which norming a_t does a non–trivial limit of $\frac{1}{a_t} Y_t$ exist in distribution as $t \to \infty$ and what is the limiting random field depends on more specific properties of the system. In this paper we investigate this question for branching particle systems in locally compact Abelian groups, where the individual particle motion is a process with symmetric stationary independent increments, and for the so called "2–level" branching systems in which not only the individual particles but also whole families of particles undergo critical branching. Multilevel branching systems were introduced by Dawson and Hochberg[9], and they have been studied by several authors: Dawson et al[10], Gorostiza[16], Gorostiza et al[17], Greven and Hochberg[24], Hochberg[25], Hochberg and Wakolbinger[26], Wu[40].

For a transient motion and finite variance branching, the simple branching particle system converges as $t \to \infty$ to a Poisson system of independently evolving "clans", each of which contributes to the occupation time. The asymptotics of the occupation time fluctuations should be determined by the space–time correlations within single clans. Also, the growth of the occupation time fluctuations as $t \to \infty$ should depend on whether there are long time dependencies caused by recurrent visits of single clans to bounded sets.

Let us first recall some known results. For a critical finite variance branching Brownian system X_t in $\mathbb{R}^d$, started off from a Poisson system with Lebesgue intensity λ, the right norming a_t for the occupation time fluctuation is $t^{3/4}$ for $d = 3$, $(t \log t)^{1/2}$ for $d = 4$, and $t^{1/2}$ for $d > 4$ (see Cox and Griffeath[5], and also Iscoe[27], where the corresponding superprocess scenario is treated). We refer to this as the 1–level branching case.

The same normings appear for the occupation time fluctuations of Poisson systems of Brownian particles without branching, which we call 0–level systems, but two dimensions lower (Cox and Griffeath[4], Deuschel and Wang[14]), and we shall see that they also appear in the 2–level branching case, but now two dimensions higher than in the 1–level case.

There is an apparent relation between the critical dimension for transience of the motion and the critical dimension for the classical $t^{1/2}$–norming of the occupation time fluctuations: for Brownian particle systems without branching, 2 is the critical dimension above which the occupation time fluctuations have

the classical norming and it is also the dimension above which the particle motion is transient. In the 1–level branching case, 4 is the critical dimension above which the occupation time fluctuations have the classical norming and it is also the dimension above which the equilibrium clans are transient (in the sense that they eventually leave each bounded region of $\mathbb{R}^d$ forever (Stöckl and Wakolbinger[38]). We shall see that an analogous result holds for the 2–level case.

One of our main objectives is to put these results in a general context for branching systems in locally compact Abelian groups, which clarifies the role played by the Green potential operator G of the particle motion and the (operator) powers G^k of G in relation with the various levels $k = 0, 1, 2, \ldots$ of branching. A key role will be played by the level k transience and recurrence properties of the motion, $k = 0, 1, 2, \ldots$ defined in Section 2. In this paper we will treat only branching levels $k = 0, 1, 2$, but the results show a pattern which allows one to guess what the form of the results would be for systems with higher levels of branching. The analysis of 2–level systems is considerably more difficult than that of 1–level systems due to the dependencies among the particles caused by the simultaneous branching of families of particles.

Roughly speaking, the bigger the branching level k is, the more long range dependencies are introduced into the system. These dependencies increase the mass fluctuations in the system, whereas a strong spreading out of mass by the particle motion has a smoothing effect on the mass fluctuations.

In the case of finite variance branching (where we will restrict for simplicity to binary branching) it turns out that finiteness of G^k corresponds to existence of the k–level branching equilibrium clans, and then $G^{k+1} = \infty$ corresponds to a long time dependence of the visits of single k–level clans to bounded sets. In the latter case, a crucial feature of these models is that the growth of the $(k+1)$–st power of the operator $G_t = \int_0^t T_s ds$ as $t \to \infty$, where T_t denotes the semigroup of the individual motion, determines the right norming a_t for the occupation time fluctuations of the k–level system, whereas for finite G^{k+1} their right norming is the classical $t^{1/2}$. In the case of finite G^{k+1} the covariance of the limiting Gaussian field of the occupation time fluctuations of the k–level system contains terms induced by direct ancestry and by level j–relationship, $1 \leq j \leq k$.

For infinite variance "$(1 + \beta)$–branching" $(0 < \beta < 1)$, we have so far results only for the case with $t^{1/(1+\beta)}$–norming. Then the norming is determined by the highest level of branching, and the occupation time fluctuations converge to stable random fields.

For the superprocess limits of the 1– and the 2–level branching particle systems the corresponding occupation time fluctuation results are basically analogous to those for the particle systems and even somewhat simpler.

We will focus on two examples of particle motions W_t: the symmetric α–stable Lévy process in $\mathbb{R}^d, 0 < \alpha \leq 2$ (including Brownian motion, $\alpha = 2$), and a "hierarchical" random walk in Ω_N, the

hierarchical group of order N, which is a direct sum of a countable number of copies of $\mathbb{Z}_{N-1}$, i.e. $\Omega_N = \{x = (x_1, x_2, \ldots) \ |x_i \in \{0, 1, \ldots, N-1\}, x_i \neq 0$ except for finitely many i$\}$.

For the symmetric α–stable process in $\mathbb{R}^d$ and $k \geq 0$, G_t^{k+1} has a power growth in t if $d/(k+1) < \alpha$, a logarithmic growth if $\alpha = d/(k+1)$, and G^{k+1} is finite if $\alpha < d/(k+1)$.

For the random walk in Ω_N we consider a probability of jumping a distance i proportional to $(c/N)^{i-1}$, where c is a constant such that $0 < c < N$. (Here, the distance between two elements (x_i) and (y_i) of Ω_N is defined to be the highest index i for which x_i and y_i are different). Since the random walk is characterized by this "mobility parameter" c, we will call it the *c–hierarchical random walk* in Ω_N. In this case, for $k \geq 0$, G_t^{k+1} has a power growth in t if $c < N^{k/(k+1)}$, a logarithmic growth if $c = N^{k/(k+1)}$, and G^{k+1} is finite if $c > N^{k/(k+1)}$. In this example the growths of G_t and its powers have also oscillating modulations in a logarithmic time scale. However, the oscillations vanish in the cases of logarithmic growth. Oscillatory phenomena have also been observed in another class of random walks on groups which are direct sums of a countable number of copies of a discrete group (Cartwright[2]), and in random walks on the Sierpiński graph (Barlow and Perkins[1], Grabner and Woess[23] and references therein). A basic reference for random walks on Abelian groups is the paper by Kesten and Spitzer[29].

The values of the parameters α (or d) and c which correspond to logarithmic growths of the powers G_t^{k+1} will sometimes be referred to as "critical", since they are boundaries between intervals with different regimes of the longtime behavior of the systems.

We will show that the asymptotics of the occupation time fluctuations of the branching systems are analogous for the two examples. A key role is played by a constant $\gamma > -1$, which is $\gamma = d/\alpha - 1$ for the α-stable system in $\mathbb{R}^d$, and $\gamma = \log c/\log(N/c)$ for the c–hierarchical system in Ω_N. For $\gamma \in (-1, 0)$, G_t grows like $t^{-\gamma}$, for $\gamma = 0$, G_t grows logarithmically, and for $\gamma > 0$, $G - G_t$ decays like $t^{-\gamma}$. Thus, γ is an order parameter for the transience/recurrence behavior of the motion, and we will refer to the three above mentioned cases as *recurrence of order $-\gamma$*, *critical recurrence*, and *transience of order γ*. It turns out that G_t^{k+1} grows like $t^{k-\gamma}$ if $k > \gamma$, grows logarithmically if $k = \gamma$, and G^{k+1} is finite if $k < \gamma$.

For the symmetric α–stable processes in $\mathbb{R}^d$, γ is restricted to $[(d-2)/2, \infty)$, whereas for the c–hierarchical random walks in Ω_N, γ can range over the entire interval $(-1, \infty)$. In this sense the hierarchical random walks are a richer class of models. Choosing $c = N^{1-\alpha/d}$, the c–hierarchical random walk in Ω_N has the same order of transience/recurrence as the α–stable process in $\mathbb{R}^d$, and this allows to think about non–integer dimensions d.

The idea of using hierarchical systems as models of systems with non–integer dimension was first introduced in the context of statistical physics. A model of ferromagnetic behavior involving the case of

$N = 2$ is known as Dyson's hierarchical model and has been used by Sinai[36] as a framework in which to carry out a rigorous renormalization group analysis following the ideas of Wilson[39]. In the case of ferromagnetism, 4 is the critical dimension and the hierarchical group has been used to study large scale fluctuations near the critical point in $4-\varepsilon$ dimensions. In the case of 1–level branching Brownian motion in $\mathbb{R}^d$, the dimension 2 is the critical dimension for the persistence/extinction dichotomy. In Dawson and Greven[7], 1–level hierarchical branching random walks (indexed by a sequence (c_j) rather than just one parameter c) have been analyzed in the so called mean–field limit $N \to \infty$ in the "nearly 2–dimensional analogue". Since the dimension 4 is the critical dimension for the occupation time fluctuations of 1–level branching Brownian motions and for the long time behavior of 2–level branching Brownian motions in $\mathbb{R}^d$, it is conceivable that the hierarchical mean–field limit can be used to carry out a similar analysis of these phenomena "near dimension 4".

The outline of the paper is as follows: Section 2 presents the general results for the particle systems, and the results for the superprocesses are mentioned. Sections 3 and 4 are devoted to the two above mentioned examples of individual motions. Section 4 contains a list of constants and functions that appear in the results of Section 3. The proofs are given in section 5. An Appendix contains definitions and background on 1– and 2–level branching systems, and some tools.

2. GENERAL NOTIONS, RESULTS AND COMMENTS

2.1. The individual motion: powers of the Green potential, strong and weak transience

We consider as a space of sites a locally compact Abelian group S with Haar measure ρ, and as individual particle motion a process W_t with stationary independent increments. We assume that for each $s > 0, W_s - W_0$ has a strictly positive symmetric density with respect to ρ, and that W_t has càdlàg paths.

Let us fix some notation. The function spaces $\mathcal{C}_c(S), \mathcal{C}_\tau(S), \mathcal{C}_c^+(S), \mathcal{C}_\tau^+(S)$, and the measure spaces $\mathcal{M}_\tau(S), \mathcal{N}_\tau(S)$, where τ is a reference function on S, are defined in the Appendix. For $\mu \in \mathcal{M}_\tau(S)$ and $\varphi \in \mathcal{C}_\tau(S)$, we write $\langle \mu, \varphi \rangle = \int \varphi d\mu$, and we denote

$$(\varphi, \psi)_\rho = \int_S \varphi\psi \, d\rho, \qquad \varphi, \psi \in \mathcal{C}_\tau(S).$$

We designate by T_t the semigroup of W_t. Note that ρ is T_t–reversible, i.e., $(\varphi, T_t\psi)_\rho = (T_t\varphi, \psi)_\rho$ for all $\varphi, \psi \in \mathcal{C}_\tau(S)$ and $t > 0$, which implies that ρ is T_t–invariant for each $t > 0$.

The *Green potential operator* G of the motion is defined by

$$G\varphi = \int_0^\infty T_t\varphi\,dt, \qquad \varphi \in \mathcal{C}_c(S).$$

Together with T_t, also G is self–adjoint with respect to ρ, so

$$\langle \rho, (G\varphi)(G\psi)\rangle = (\varphi, G^2\psi)_\rho.$$

Observe that the (operator) powers of G are given by

$$G^k\varphi = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}T_t\varphi\,dt, \quad k \geq 2. \tag{2.1.1}$$

The quantities $G^{k+1}(x,B) := G^{k+1}1_B(x), x \in S, B$ a measurable subset of S, $k = 0,1,...,$ have an interpretation in terms of occupation times of a mass flow with continuous birth of mass, which will be helpful later on in the genealogical picture of 1– and 2–level branching systems. Consider the case $k = 1$ and imagine an initial "parent" unit mass at $x \in S$, which evolves according to the flow T_t. This parent mass generates its own amount of "daughter" mass at its own site continuously at rate 1, and this daughter mass is again transported by the flow T_t. Then $G^2(x,B)$ is simply the total occupation time of the daughter mass in B; this is immediate from the semigroup property. To interpret $G^{k+1}(x,B)$ in a similar way, imagine an initial "k–level" unit mass at x which evolves according to the flow T_t. For every $j = k,\ldots,1$, the j–level mass generates its own amount of $(j-1)$–level mass, which is again transported by the flow T_t. Then $G^{k+1}(x,B)$ results as the total occupation time of 0–level mass in B.

We use the notation $||\cdot||$ for the supremum norm.

Definition 2.1.1. (a) For $k \geq 0$, we say that W_t is *level k transient* if

$$||G^{k+1}\varphi|| < \infty \;\; \text{for}\, \varphi \in \mathcal{C}_c^+(S),$$

and *level k recurrent* if

$$G^{k+1}\varphi \equiv \infty \text{ for } \varphi \in \mathcal{C}_c^+(S),\, \varphi \neq 0.$$

(b) For $k \geq 0$, we say that W_t is *level k strongly transient* if it is level $k+1$ transient, and W_t is *level k weakly transient* if it is level k transient and level $k+1$ recurrent.

Note that level 0 transience and recurrence are (because of the assumed irreducibility of W_t) just the ordinary notions of transience and recurrence, and note also that level 0 strong and weak transience coincide with the notions of strong and weak transience as defined, e.g., in Port and Stone[(33)]. Clearly, level k transience implies level j transience for $j < k$, and level k recurrence implies level j recurrence for $j > k$. In terms of the interpretation given above, level k transience (resp. recurrence) means that

a k–level parent unit mass sends up to time infinity a finite (resp. infinite) amount of 0–level daughter mass to any bounded region.

Definition 2.1.2. (a) We define the operator

$$G_t\varphi = \int_0^t T_s\varphi\, ds, \quad \varphi \in \mathcal{C}_\tau(S), \quad t > 0,$$

and denote by G_t^k the k–th (operator) power of G_t, $k \geq 2$.

(b) For transient motion, we define the the bilinear form

$$R_t(\varphi,\psi) = (\varphi, (G - G_t)\psi)_\rho, \quad \varphi,\psi \in \mathcal{C}_\tau(S), \quad t > 0.$$

Note that for each $k \geq 1$ and $\varphi \in \mathcal{C}_c^+(S), \varphi \neq 0$, $\int_0^\infty t^{k-1}R_t(\varphi,\varphi)dt < \infty$ (resp. $= \infty$) if the motion is level k transient (resp. level k recurrent).

Definition 2.1.3. (a) Let H and, for each $t > 0, H_t$ be positive definite bilinear forms on $\mathcal{C}_c(S)$, and let $f : [T,\infty) \to \mathbb{R}_+$ for some $T > 0$. We write $H_t \sim f_t H$ if

$$\frac{1}{f_t} H_t(\varphi,\psi) \to H(\varphi,\psi) \quad \text{as} \quad t \to \infty, \ \varphi,\psi \in \mathcal{C}_c^+(S).$$

If Q_t is a linear operator from $\mathcal{C}_c(S)$ into $\mathcal{C}_\tau(S)$, the notation $Q_t \sim f_t H$ means that $H_t \sim f_t H$ holds for $H_t(\varphi,\psi) := (\varphi, Q_t\psi)_\rho$.

(b) We call f_t a *growth function* if it is increasing and $\lim_{t\to\infty} f_t = \infty$.

Growth functions will be used to characterize the growth of G_t and its powers. The growth functions we shall encounter in the examples are of the form $f_t = t^\zeta h_t$, where $\zeta \in (0,1)$ and h_t is either identically equal to 1 or a slowly oscillating function, or $f_t = \log t$. Part (a) of the definition will also be used to characterize the "order of transience" of W_t in terms of R_t with $f_t \to 0$.

2.2. Main results: Occupation time fluctuations for k–level critical binary branching particle systems ($k = 0, 1, 2$)

We consider the following particle systems in S with the individual particles moving independently according to the process W_t:

(i) 0–*level system*: The system starts off from a Poisson system with intensity ρ.

(ii) 1–*level branching system*: The motion is transient, the system starts off from a Poisson system with intensity ρ, and the particles undergo critical binary branching at rate V. We recall that this system

has a (infinitely divisible) "Poisson type" equilibrium (in the sense of Liemant et al[31], section 2.3), and we denote by R^1_∞ its canonical measure, which we call *equilibrium canonical measure.* Note that R^1_∞ has intensity ρ (see the Appendix, (A.1.11)).

(iii) 2*–level branching system*: The motion is strongly transient, the system starts off from a Poisson system of "2–level particles" with intensity R^1_∞, individual particles undergo critical binary branching at rate V_1 and clans undergo critical binary branching at rate V_2. Note that R^1_∞ is an invariant measure for the 1–level dynamics, just as ρ is an invariant measure for the 0–level dynamics (however, R^1_∞ is not reversible for the 1–level dynamics).

We refer the reader to Gorostiza[16], and Hochberg and Wakolbinger[26] for a detailed description of the dynamics of 2–level branching systems. The necessary background for the present paper is given in the Appendix and should be consulted as the need arises.

For the three particle systems above, X_t stands for the empirical measure of the locations of all the particles present at time t. Thus, X_t is a random point measure on S. In the 2–level case X_t corresponds to the *aggregated* system, i.e., the particles are counted as "1–level particles" independently of what clans ("2–level particles") they belong to. In each one of these systems the intensity is preserved, i.e., $EX_t = \rho$ for all $t > 0$, as a consequence of the initial conditions and the criticality in the branching cases. We consider the *occupation time fluctuation*, which is the random signed measure Y_t on S defined by

$$Y_t = \int_0^t (X_s - \rho) ds, \quad t > 0.$$

The following theorems describe the asymptotic distribution of Y_t as $t \to \infty$ for the k–level systems, $k = 0, 1, 2$, described above. All the convergence assertions are understood to be in distribution as $t \to \infty$, all the test functions belong to $\mathcal{C}_c^+(S)$, i.e., all random fields are considered over $\mathcal{C}_c^+(S)$. The results for the 0–level system are basically known in special cases, but we include them for completeness and because they are the initial step in the multilevel ladder.

Theorem 2.2.1. Let X_t be the 0–level system.

(a) If the motion W_t is transient, then $t^{-1/2} Y_t$ converges to a Gaussian field with covariance functional $2(\varphi, G\psi)_\rho$.

(b) If the motion W_t is recurrent with $G_t \sim f_t H$ for some growth function f_t, then $(\int_0^t f_s \, ds)^{-1/2} Y_t$ converges to a Gaussian field with covariance functional $2H(\varphi, \psi)$.

Theorem 2.2.2. Let X_t be the 1–level branching system.

(a) If W_t is strongly transient, then $t^{-1/2}Y_t$ converges to a Gaussian field with covariance functional

$$(\varphi, (2G + VG^2)\psi)_\rho = 2\left(\varphi, \left(I + \frac{V}{2}G\right) G\psi\right)_\rho .$$

(b) If W_t is weakly transient with $G_t^2 \sim f_t H$ for some growth function f_t, then $(\int_0^t f_s\, ds)^{-1/2}Y_t$ converges to a Gaussian field with covariance functional $VH(\varphi, \psi)$.

Theorem 2.2.3. Let X_t be the 2–level branching system.

(a) If W_t is level 1 strongly transient and if there exists $\delta > 5/2$ such that

$$||T_t\varphi|| = O(t^{-\delta}) \quad \text{as} \quad t \to \infty,\ \varphi \in \mathcal{C}_c^+(S), \tag{2.2.1}$$

then $t^{-1/2}Y_t$ converges to a Gaussian field with covariance functional

$$\left(\varphi, \left(2G + (V_1 + V_2)G^2 + \frac{1}{2}V_1V_2G^3\right)\psi\right)_\rho = 2\left(\varphi, \left(I + \frac{V_2}{2}G\right)\left(I + \frac{V_1}{2}G\right) G\psi\right)_\rho .$$

(b) If W_t is level 1 weakly transient with $G_t^2 G \sim f_t H$ for some growth function f_t, then $(\int_0^t f_s\, ds)^{-1/2}Y_t$ converges to a Gaussian field with covariance functional $\frac{1}{2}V_1V_2H(\varphi, \psi)$.

2.3. Comments on the assumptions and the results

1. In Subsection 2.4 we will give conditions on the motion W_t which imply the growth assumptions of Theorems 2.2.2(b) and 2.2.3(b). The α–stable motion fits into this framework.

2. We do not know if condition (2.2.1) holds in general for level 1 strongly transient motion. We will show, however, that it holds for the motions in the examples.

3. We give an explanation of the second moment structure ocurring in Theorem 2.2.2(a). Let $\underline{R}^1_\infty$ be the historical counterpart of the canonical equilibrium measure R^1_∞ (which is a time–shift invariant measure on the space of clans ranging from time $-\infty$ to time $+\infty$) (see Dawson and Perkins[12], Sections 5.4.3, 5.4.4). Firstly, we observe that the second moment measure of R^1_∞ is given by (see the Appendix, (A.1.12))

$$\int \langle \mu, \varphi\rangle\langle \mu, \psi\rangle R^1_\infty(d\mu) = \left(\varphi, \left(I + \frac{1}{2}VG\right)\psi\right)_\rho . \tag{2.3.1}$$

The equality (2.3.1) can be intuitively understood through the backward tree picture (Gorostiza and Wakolbinger[22], Section 4, and references therein, and Appendix, Section 3.A): The intensity

measure of the canonical Palm distribution $(R^1_\infty)_x$ (with *ego* at site x) is

$$\begin{aligned}\int \langle \mu, \psi\rangle (R^1_\infty)_x(d\mu) &= \psi(x) + E_x\left(\int_0^\infty \int p_t(W_t, dy)\psi(y)V\,dt\right)\\ &= \psi(x) + \frac{1}{2}VG\psi(x), \qquad (2.3.2)\end{aligned}$$

where p_t denotes the transition probability of the motion. Then, by the Palm formula (A.2.1) (Appendix), (2.3.1) follows by integrating (2.3.2) with respect to $\varphi(x)\rho(dx)$. The same reasoning shows that the space–time correlation structure of $\underline{R}^1_\infty$ is given by

$$E_{\underline{R}^1_\infty}(\langle X_t, \varphi\rangle\langle X_{t+s}, \psi\rangle) = \left(\varphi, \left(I + \frac{1}{2}VG\right)T_s\psi\right)_\rho.$$

This reveals how the normed second moment measure of the occupation time behaves:

$$\frac{1}{t}E_{\underline{R}^1_\infty}\left(\int_0^t \langle X_s, \varphi\rangle ds \int_0^t \langle X_r, \psi\rangle dr\right) \to 2\left(\varphi, \left(I + \frac{V}{2}G\right)G\psi\right)_\rho \quad \text{as } t \to \infty.$$

4. Because of the obvious identities

$$\frac{1}{2}(\varphi, G\varphi)_\rho = \int_0^\infty (\varphi, T_{2r}\varphi)_\rho\,dr = \int_0^\infty \langle \rho, (T_r\varphi)^2\rangle dr,$$

transience of the motion implies persistence of the 1–level branching system (Gorostiza and Wakolbinger(19) Corollary 2.2). Therefore the existence of the measure R^1_∞, which is the assumed intensity for the Poisson initial condition of the 2–level branching system, is implied by the strong transience assumption on the system.

5. We now explain the second moment structure appearing in Theorem 2.2.3(a). For a level 1 strongly transient motion it can be shown along the same lines as in Gorostiza et al(17) that the 2–level branching particle system, started off from the Poisson system of 1–level equilibrium clans, is persistent. Using e.g. the argument of Gorostiza(16) (Lemma 4.6), one derives the following expression for the second moment measure of the canonical measure Q_∞ of the aggregated 2–level equilibrium with intensity ρ (using the fact that R^∞_1 is the Poissonization of the canonical measure of the superprocess counterpart, Appendix, (A.1.10)):

$$\int \langle \mu, \varphi\rangle\langle \mu, \psi\rangle Q_\infty(d\mu) = \varphi, \left(I + \frac{1}{2}(V_1 + V_2)G + \frac{1}{4}V_1V_2G^2\psi\right)_\rho. \qquad (2.3.3)$$

Then the second moment structure of the occupation time follows as in the 1–level case. Equality (2.3.3) can also be understood through the backward tree picture: The intensity measure of the Palm distribution $(Q_\infty)_x$ has the representation (Hochberg and Wakolbinger(26))

$$\int \langle \mu, \psi\rangle (Q_\infty)_x(d\mu) = \psi(x) + E_x\left[\int_0^\infty \left(\int (V_1 + V_2)p_t(W_t, dy)\psi(y)\right.\right.$$

$$
\begin{aligned}
& \quad + \int_0^t \int V_1 p_s(W_t, dz) \int V_2 p_{t-s}(z, dy)\psi(y)\, ds \Big)\, dt \Big] \\
= & \quad \psi(x) + \frac{1}{2}(V_1 + V_2)G\psi(x) + \frac{1}{4}V_1V_2G^2\psi(x). \qquad (2.3.4)
\end{aligned}
$$

The summands on the r.h.s. of (2.3.4) have an interpretation in terms of the genealogy: $\frac{1}{2}V_1G\psi(x)$ is the contribution of the 1–level relatives, $\frac{1}{2}V_2G\psi(x)$ is that of the 2–level relatives breaking off directly from the individual trunk, and $\frac{1}{4}V_1V_2G^2\psi(x)$ is that of the 2–level relatives having been generated by 1–level relatives, after those have broken off from the individual trunk. Now (2.3.3) follows by integrating (2.3.4) with respect to $\varphi(x)\rho(dx)$.

6. The following table subsumes the covariance kernels appearing in the second moment structures discussed above (Theorems 2.2.2(a) and 2.2.3(a)). Columns 1, 2 and 3 refer to simple motion, 1–level branching and 2–level branching, respectively.

$2G$	$2G + VG^2$	$2G + (V_1 + V_2)G^2 + \frac{1}{2}V_1V_2G^3$
	$= 2\left(I + \frac{V}{2}G\right)G$	$= 2\left(I + \frac{V_2}{2}G\right)\left(I + \frac{V_1}{2}G\right)G$

We observe a relationship between the covariance kernels for the 1– and 2–level cases: For $V > 0$ and an operator Q, we define the operator

$$
C_V(Q) = \left(I + \frac{V}{2}G\right)Q.
$$

Then the 1– and 2–level covariance kernels are given by $2C_V(G)$ and $2C_{V_2}(C_{V_1}(G))$, respectively. Thus, the 2–level covariance kernel is like the 1–level covariance kernel with V replaced by V_2 and the operator G replaced by $C_{V_1}(G)$. Recall that G represents the expected total occupation time of the motion, and note that $C_V(G)$ represents the expected total occupation time of the mass flow of the motion plus a continuous throwing off of mass with intensity $\frac{1}{2}V$, which also evolves by the same flow. One can then guess that for a 3–level system the covariance kernel would be given by

$$
2C_{V_3}(C_{V_2}(C_{V_1}(G))) = 2\left(I + \frac{V_3}{2}G\right)\left(I + \frac{V_2}{2}G\right)\left(I + \frac{V_1}{2}G\right)G,
$$

and so on for higher levels of branching.

7. By using an argument of Dawson and Perkins[(12)] (Section 5.4.4), one observes that a level 1 transient motion leads to transient equilibrium clans. Indeed, even the expected value a of the total future occupation time in a bounded region B, starting from an equilibrium Palm cluster, is finite: Recall

from (2.3.2) that the intensity measure $\nu(dy)$ of the Palm distribution $(R^1_\infty)_0$ of an equilibrium cluster (with *ego* at the origin) is

$$\nu(dy) = \delta_0(dy) + \frac{1}{2}VG(0, dy),$$

hence

$$a = \int \nu(dy)G(y, B) = G(0, B) + \frac{1}{2}VG^2(0, B) < \infty.$$

8. A level 2 transient motion leads to transient aggregated 2–level equilibrium clans. Indeed, by (2.3.4) the intensity measure $\sigma(dy)$ of the Palm distribution $(Q_\infty)_0$ is

$$\sigma(dy) = \delta_0(dy) + \left(\frac{1}{2}(V_1 + V_2)G + \frac{1}{4}V_1V_2G^2\right)(0, dy),$$

hence the expected value of the total future occupation time in a bounded region B is

$$\int \sigma(dy)G(y, B) = G(0, B) + \left(\frac{1}{2}(V_1 + V_2)G^2 + \frac{1}{4}V_1V_2G^3\right)(0, B) < \infty.$$

9. For the 2–level system, if the intensity of the initial Poisson distribution is $\delta_{\delta_x}\rho(dx)$ (instead of R^1_∞), then in the assumption for Theorem 2.2.3(b) the growth of G^2_tG is replaced by the growth of G^3_t, and an analogous result holds.

2.4. Order of transience and recurrence, asymptotics of powers of G_t, and some special growth functions

Definition 2.4.1. (a) Let $\mathcal{H}$ denote the class of differentiable functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ that are bounded and bounded away from 0 on $[T, \infty)$ for some $T > 0$.

(b) For fixed $a \in (0, 1)$, let

$$\widetilde{\mathcal{H}}_a = \{h \in \mathcal{H} \mid h_t = h_{at} \quad \text{for all} \quad t > 0\}$$

(with $T = 0$). Note that the elements of $\widetilde{\mathcal{H}}_a$ are periodic in a logarithmic scale.

Definition 2.4.2. (a) For a given $\gamma > 0$, we say that W_t is *transient of order* γ if $R_t \sim t^{-\gamma}h_tJ$ for some $h \in \mathcal{H}$ as in Definition 2.4.1 and some bilinear form J as in Definition 2.1.3.
(b) For a given $\gamma \in (-1, 0)$, we say that W_t is *recurrent of order* $-\gamma$ if $G_t \sim t^{-\gamma}h_tJ$ for some h and J as above, and we say that W_t is *critically recurrent* if $G_t \sim \log t \cdot J$ for some J as above.

Clearly, for transience of order γ we have

$$\text{level } k \text{ transience if and only if } \; k < \gamma,$$

$$\text{level } k \text{ recurrence if and only if } \ k \geq \gamma,$$

and therefore the following lemma holds.

Lemma 2.4.1. If W_t is transient of order γ, then for each $k \geq 0, W_t$ is

$$\text{level } k \text{ strongly transient if and only if } \ \gamma > k+1,$$

$$\text{level } \ k \ \text{ weakly transient if and only if } \ k < \gamma \leq k+1.$$

The next lemma shows how the assumptions of Theorems 2.2.2(b) and 2.2.3(b) on the growth of powers of G_t are implied by transience of order γ with $h \equiv 1$.

Lemma 2.4.2. Let W_t be transient of order γ with $R_t \sim t^{-\gamma} J$.
(1) If $0 < \gamma < 1$, then

$$G_t G \sim t^{1-\gamma} H \quad \text{with} \quad H = \frac{1}{1-\gamma} J,$$

and

$$G_t^2 \sim t^{1-\gamma} H \quad \text{with} \quad H = \frac{2 - 2^{1-\gamma}}{1-\gamma} J.$$

(2) If $\gamma = 1$, then

$$G_t G \sim G_t^2 \sim \log t \cdot H \quad \text{with} \quad H = J.$$

(3) If $1 < \gamma < 2$, then

$$G_t^2 G \sim t^{2-\gamma} H \quad \text{with} \quad H = \frac{2 - 2^{2-\gamma}}{(2-\gamma)(\gamma-1)} J,$$

and

$$G_t^3 \sim t^{2-\gamma} H \quad \text{with} \quad H = \frac{3^{2-\gamma} - 2^{2-\gamma} - 1}{(2-\gamma)(\gamma-1)} J.$$

(4) If $\gamma = 2$, then

$$G_t^2 G \sim G_t^3 \sim \log t \cdot H \quad \text{with} \quad H = J.$$

More generally, under the assumptions of Lemma 2.4.2 one can show that

$$G_t^{\gamma} G \sim G_t^{\gamma+1} \sim \log t \cdot J \quad \text{if} \quad \gamma \text{ is an integer,}$$

and

$$G_t^{[\gamma]+1} G \sim t^{[\gamma]+1-\gamma} c_\gamma J, \quad G_t^{[\gamma]+2} \sim t^{[\gamma]+1-\gamma} c'_\gamma J \quad \text{otherwise}$$

(for suitable constants c_γ, c'_γ). But we will use only the cases $\gamma \leq 2$ considered in the lemma.

For growth functions f_t of the form $f_t = t^\zeta h_t, 0 < \zeta < 1, h \in \widetilde{\mathcal{H}}_a$,which will occur in one of the examples, the normalizations $(\int_0^t f_s ds)^{1/2}$ that appear in the conclusions of Theorems 2.2.1(b), 2.2.2(b) and 2.2.3(b) can be replaced by $(t^{\zeta+1}\widetilde{h}_t)^{1/2}$, where $\widetilde{h} \in \widetilde{\mathcal{H}}_a$, thanks to the following lemma.

Lemma 2.4.3. Let $h \in \widetilde{\mathcal{H}}_a$. If $\zeta > -1$, then

$$\int_1^t s^\zeta h_s ds \sim t^{\zeta+1}\widetilde{h}_t \quad \text{as}\, t \to \infty, \tag{2.4.1}$$

where

$$\widetilde{h}_t = -\log a \cdot \int_0^\infty a^{r(1+\zeta)} h_{a^r t} dr.$$

Note that $\widetilde{h} \in \widetilde{\mathcal{H}}_a$.

Lemma 2.4.3 applies to the growths of the fluctuations of the 0, 1 and 2–level hierarchical system, but the l.h.s. of (2.4.1) can be computed explicitly in this case.

2.5. Infinite variance branching

In the case of infinite variance branching we consider here only the so called "$(1+\beta)$–branching", $(0 < \beta < 1)$, whose offspring generating function is of the form $s + q(1-s)^{1+\beta}$, $s \in [0,1]$, for some constant $q > 0$ (this law belongs to the domain of normal attraction of a stable law with exponent $1+\beta$). The picture now is less complete: we have only results for the "classical" $t^{1/(1+\beta)}$–norming. In the 1–level case it can be shown (along the lines of Stöckl and Wakolbinger(38)) that this regime coincides with that of clan transience. We conjecture that an analogous result holds also in the 2–level case. Note that for $\beta = 1$ we have the binary branching system considered above. However, we shall see that the results for the finite variance case are not special cases of the ones in this subsection.

We consider the following branching systems:

(i) 1–*level system*: The system is as described in Subsection 2.2, except that the particles undergo $(1+\beta)$–branching at rate V. We assume that is is persistent. Hence the system (with Poisson initial condition) converges to an equilibrium with intensity ρ. (Sufficient conditions for this persistence are given in Gorostiza and Wakolbinger(22), Theorem 2.1). The equilibrium is then a Poisson superposition of "2–level particles". Again we denote the equilibrium canonical measure by R^1_∞.

(ii) 2–*level system*: The system is as described in Subsection 2.2, except that individual particles

undergo $(1+\beta_1)$–branching at rate V_1, clans undergo $(1+\beta_2)$–branching at rate V_2, and it starts off from a Poisson system of "2–level particles" with intensity R^1_∞ (the equilibrium canonical measure for the 1–level $(1+\beta_1)$–branching system with rate V_1).

As above, Y_t denotes the occupation time fluctuation (of the aggregated system in the 2–level case).

Theorem 2.5.1. Let X_t be the 1–level branching system. Assume that

$$\langle \rho, (G\varphi)^{1+\beta}\rangle < \infty, \quad \varphi \in \mathcal{C}^+_c(S). \tag{2.5.1}$$

Then $t^{-1/(1+\beta)}Y_t$ converges to a random field Z with Laplace functional

$$E\exp\{-\langle Z, \varphi\rangle\} = \exp\left\{\frac{V}{1+\beta}\langle \rho, (G\varphi)^{1+\beta}\rangle\right\}, \quad \varphi \in \mathcal{C}^+_c(S).$$

Here and in the next theorem the notation $\langle Z, \varphi\rangle$ means the action of the random field Z on φ.

Note that the finite variance case $\beta = 1$ (Theorem 2.2.2 (a)) is not a special case of Theorem 2.5.1, since this theorem would provide only the second term of the covariance functional. The term $2(\varphi, G\psi)_\rho$ in Theorem 2.2.2(a) does not appear in Theorem 2.5.1 because the normalization is now strong enough to kill this contribution to the occupation time fluctuations which comes from individuals related in direct line.

Theorem 2.5.2. Let X_t be the 2–level branching system. Assume that $\beta_2 < 1$ and

$$\int \langle \mu, G\varphi\rangle^{1+\beta_2} R^1_\infty(d\mu) < \infty, \quad \varphi \in \mathcal{C}^+_c(S). \tag{2.5.2}$$

Then $t^{-1/(1+\beta_2)}Y_t$ converges to a random field Z with Laplace functional

$$E\exp\{-\langle Z, \varphi\rangle\} = \exp\left\{\frac{V_2}{1+\beta_2}\int \langle \mu, G\varphi\rangle^{1+\beta_2} R^1_\infty(d\mu)\right\}, \quad \varphi \in \mathcal{C}^+_c(S).$$

Note that also for the 2–level system the finite variance case $\beta_1 = \beta_2 = 1$ (Theorem 2.2.3(a)) is not a special case of Theorem 2.5.2.

In the case of α–stable motions in $\mathbb{R}^d$, we shall see in the next section that conditions (2.5.1) and (2.5.2) hold for "high" dimensions.

2.6. Occupation time fluctuations of superprocesses

As stated in the Introduction, results analogous to the previous ones hold also for the occupation time fluctuations of the 1– and 2– level superprocesses corresponding to the branching particle systems.

The results are simpler because all the mass relationships closer than level k do not contribute to the occupation time fluctuation limit of the k–level branching system. Theorems 2.2.2(a) and 2.2.3(a) hold with only the terms involving G^2 and G^3 present in the covariances of the limit random fields, respectively. The proofs are similar to those for the branching particle system and we shall omit them.

3. TWO EXAMPLES OF INDIVIDUAL MOTIONS: SYMMETRIC α–STABLE PROCESS AND c–HIERARCHICAL RANDOM WALK

3.1. Transience/recurrence properties of the motions

In the first example the particle motion W_t is the spherically symmetric α–stable Lévy process in $S = \mathbb{R}^d$, $(0 < \alpha \leq 2)$, and ρ is the Lebesgue measure λ. In the second example W_t is a continuous–time random walk in the hierarchical group $S = \Omega_N$ and ρ is the counting measure ν. Due to similarities in the asymptotic behavior of G_t and its powers for the two examples, which we will work out in this subsection, the limits of the occupation time fluctuations of the corresponding 1– and 2–level branching systems will also be analogous.

In this subsection we use several constants and functions whose definitions are collected in Section 4 for easy reference.

The analogy between the two motions is exhibited by a constant γ which we will define in each case.

For the α–stable process we define

$$\gamma = \frac{d}{\alpha} - 1. \tag{3.1.1}$$

Before defining the γ for the hierarchical random walk we will give some background.

The *hierarchical group* of order N is a countable group defined by

$$\Omega_N = \{x = (x_1, x_2, \ldots) \mid x_i \in \{0, 1, \ldots, N-1\}, x_i \neq 0 \text{ except for finitely many } i\},$$

with addition componentwise mod(N). The *hierarchical distance* $|\cdot|$ on Ω_N is defined by

$$|x - y| = \max\{i | x_i \neq y_i\}.$$

A discrete–time *hierarchical random walk* in Ω_N jumps from x to y such that $|x - y| = i \geq 1$ with probability $r_i/N^{i-1}(N-1)$, where $r_1, r_2, \ldots$ is a probability distribution on $\{1, 2, \ldots\}$. This type of random walk was introduced by Spitzer[37] for $N = 2$ (the "light bulb random walk"), and by Sawyer and Felsenstein[35] for general N in the context of genetics models.

The continuous–time analogue of the hierarchical random walk with jump rate $\sigma > 0$ has transition density

$$p_t(0,x) = \frac{1}{N^i}(\delta_{0i} - 1)e^{-\sigma(1-f_i)t} + (N-1)\sum_{j=i+1}^{\infty} \frac{1}{N^j} e^{-\sigma(1-f_j)t}$$

if $|x| = i \geq 0$, where f_j is given by

$$f_j = r_1 + \ldots + r_{j-1} - \frac{r_j}{N-1}, \quad j \geq 1,$$

(note that f_0 is irrelevant); see e.g. Fleischmann and Greven[15] for additional information.

Here we will take r_i of the form

$$r_i = \left(\frac{c}{N}\right)^{i-1}\left(1 - \frac{c}{N}\right), \quad i \geq 1,$$

where c is a constant such that $0 < c < N$. In this case we have $f_j = 1 - a^j b, \quad j \geq 1$, where

$$a = \frac{c}{N}, \quad b = \frac{N^2/c - 1}{N-1}, \tag{3.1.2}$$

(note that $0 < a < 1, b > 1$), and the transition density becomes

$$p_t(0,x) = \frac{1}{N^i}(\delta_{0i} - 1)e^{-\sigma a^i bt} + (N-1)\sum_{j=i+1}^{\infty} \frac{1}{N^j} e^{-\sigma a^j bt} \tag{3.1.3}$$

if $|x| = i \geq 0$.

Since this random walk is characterized by the constant c (for fixed N), we will call it the *c–hierarchical random walk*. We shall see that c is a mobility constant which plays a similar role to that of α for the α-stable process.

We define

$$\gamma = \frac{\log c}{\log(N/c)}, \quad \textit{i.e.}, \quad c = N^{\gamma/(\gamma+1)}, \tag{3.1.4}$$

The following lemma shows that the constant γ defined in (3.1.1) for the α–stable process and in (3.1.4) for the c–hierarchical random walk corresponds to the order of transience/recurrence parameter (Definition 2.4.2) for the respective processes, and it also shows the analogies for the semigroups T_t and for the growth conditions for G_t^2 and $G_t^2 G$ that appear as assumptions in Theorem 2.2.2(b) and Theorem 2.2.3(b) for the two motions. Each one of the functions h_t appearing in the lemma equals 1 for the α-stable process, and belongs to $\widetilde{\mathcal{H}}_a$ for the c-hierarchical random walk, with a given by (3.1.2), or, equivalently, by (4.2.3).

Lemma 3.1.1. Let W_t be either the α–stable process in $\mathbb{R}^d$ or the c–hierarchical random walk in Ω_N.

(a) For all $\gamma > -1$,

$$(\varphi, T_t\psi)_\rho \sim t^{-(\gamma+1)} h_t q_\gamma \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(b) For $\gamma > 0, W_t$ is transient of order γ with

$$R_t(\varphi,\psi) \sim t^{-\gamma} h_t q_\gamma \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(c) For $-1 < \gamma < 0, W_t$ is recurrent of order $-\gamma$ with

$$(\varphi, G_t\psi)_\rho \sim t^{-\gamma} h_t q_\gamma \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(d) For $\gamma = 0, W_t$ is critically recurrent with

$$(\varphi, G_t\psi)_\rho \sim \log t \cdot q_0 \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(e) For $0 < \gamma < 1, W_t$ is weakly transient with

$$(\varphi, G_t^2\psi)_\rho \sim t^{1-\gamma} h_t q_\gamma \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(f) For $\gamma = 1, W_t$ is weakly transient with

$$(\varphi, G_t^2\psi)_\rho \sim \log t \cdot q_1 \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(g) For $1 < \gamma < 2, W_t$ is level 1 weakly transient with

$$(\varphi, G_t^2 G\psi)_\rho \sim t^{2-\gamma} h_t q_\gamma \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

(h) For $\gamma = 2, W_t$ is level 1 weakly transient with

$$(\varphi, G_t^2 G\psi)_\rho \sim \log t \cdot q_2 \langle\rho,\varphi\rangle\langle\rho,\psi\rangle.$$

The correspondences for the examples are:
For the α–stable process: $\gamma = \dfrac{d}{\alpha} - 1, \rho = \lambda, h \equiv 1$ in all cases,

(a) $q_\gamma = \kappa_{d,\gamma}$,

(b) $\alpha < d, q_\gamma = \dfrac{\kappa_{d,\gamma}}{\gamma}$,

(c) $\alpha > d, q_\gamma = \dfrac{\kappa_{d,\gamma}}{-\gamma}$,

(d) $\alpha = d, q_0 = \kappa_{d,0}$,

(e) $\frac{d}{2} < \alpha < d, q_\gamma = \frac{2-2^{1-\gamma}}{(1-\gamma)\gamma}\kappa_{d,\gamma}$,

(f) $\alpha = \frac{d}{2}, q_1 = \kappa_{d,1}$,

(g) $\frac{d}{3} < \alpha < \frac{d}{2}, q_\gamma = \frac{2-2^{2-\gamma}}{(2-\gamma)(\gamma-1)\gamma}\kappa_{d,\gamma}$,

(h) $\alpha = \frac{d}{3}, q_2 = \kappa_{d,2}$.

For the c–hierarchical random walk: $\gamma = \frac{\log c}{\log(N/c)}, \rho = \nu$,

(a) $q_\gamma = \kappa_{N,\gamma}, h = h^{(1,\gamma+1)}$,

(b) $c > 1, q_\gamma = \kappa_{N,1}, h = h^{(1,\gamma)}$,

(c) $c < 1, q_\gamma = \kappa_{N,\gamma}, h = h^{(2,\gamma)}$,

(d) $c = 1, q_0 = \frac{\kappa_{N,0}}{\log N}$,

(e) $1 < c < N^{1/2}, q_\gamma = \kappa_{N,\gamma}, h = h^{(3,\gamma-1)}$,

(f) $c = N^{1/2}, q_1 = \frac{2\kappa_{N,1}}{\log N}$,

(g) $N^{1/2} < c < N^{2/3}, q_\gamma = \kappa_{N,\gamma}, h = h^{(3,\gamma-2)}$,

(h) $c = N^{2/3}, q_2 = \frac{3\kappa_{N,2}}{\log N}$.

The constants $\kappa_{d,\alpha}$ and $\kappa_{N,\gamma}$, and the functions $h^{(\cdot,\cdot)}$ are defined in Section 4: expressions (4.1.1), (4.2.1), (4.2.5), (4.2.6) and (4.2.7).

Corollary 3.1.1 (to Lemmas 3.1.1 and 2.4.1).

The α–stable process is level k strongly transient if and only if

$$\alpha < \frac{d}{k+2},$$

and level k weakly transient if and only if

$$\frac{d}{k+2} \leq \alpha < \frac{d}{k+1}.$$

The c–hierarchical random walk is level k strongly transient if and only if

$$c > N^{(k+1)/(k+2)},$$

and level k weakly transient if and only if

$$N^{k/(k+1)} < c \leq N^{(k+1)/(k+2)}.$$

Corollary 3.1.1 for the α–stable process with $k = 0$ is well known (Sato[34]).

The powers of the Green potential operator of W_t are given in the next lemma.

Lemma 3.1.2. Let $\gamma > 0$ and $1 \leq j < \gamma + 1$.

(a) For the α–stable process, the integral kernel of G^j is $G_{d,\gamma,j}(x)$ given by (4.1.2) with $\alpha j < d$.

(b) For the c–hierarchical random walk, the integral kernel of G^j is $G_{N,\gamma,j}(x)$ given by (4.2.2) with $c > N^{(j-1)/j}$.

The next lemma shows that the condition (2.2.1) in Theorem 2.2.3(a) is fulfilled in both examples with $\delta = 3$.

Lemma 3.1.3. Let W_t be either the α-stable process in $\mathbb{R}^d$ or the c–hierarchical random walk in Ω_N. Then level k strong transience implies that $||T_t\varphi|| = \mathrm{o}(t^{-(k+2)})$, $\varphi \in \mathcal{C}_c^+(\mathbb{R}^d)$ (resp. $C_c^+(\Omega_N)$).

We give next upper and lower bounds for the functions h and $\widetilde{h}$ defined in Section 4, which appear in Lemma 3.1.1 and in Theorem 3.2.1 below for the hierarchical case.

Proposition 3.1.1.

Function	Lower bound	Upper bound
$h_t^{(1,\zeta)}, \zeta > 0$	$\dfrac{\Gamma}{a^{-\zeta} - 1}$	$\dfrac{a^{-\zeta}\Gamma}{a^{-\zeta} - 1}$
$h_t^{(2,\zeta)}, -1 < \zeta < 0$	$\dfrac{a^{-\zeta}\Gamma}{1 - a^{-\zeta}}$	$\dfrac{\Gamma}{1 - a^{-\zeta}}$
$h_t^{(3,\zeta)}, -1 < \zeta < 0$	$\dfrac{a^{-\zeta}(2 - 2^{-\zeta})\Gamma}{1 - a^{-\zeta}}$	$\dfrac{(2 - 2^{-\zeta})\Gamma}{1 - a^{-\zeta}}$
$\widetilde{h}_t^{(2,\zeta)}, -1 < \zeta < 0$	$\dfrac{a^{-\zeta}\Gamma}{(1 - a^{-\zeta})(1 - \zeta)}$	$\dfrac{\Gamma}{(1 - a^{-\zeta})(1 - \zeta)}$
$\widetilde{h}_t^{(3,\zeta)}, -1 < \zeta < 0$	$\dfrac{a^{-\zeta}(2 - 2^{-\zeta})\Gamma}{(1 - a^{-\zeta})(1 - \zeta)}$	$\dfrac{(2 - 2^{-\zeta})\Gamma}{(1 - a^{-\zeta})(1 - \zeta)}$

where $\Gamma \equiv \Gamma(\zeta + 1)$.

3.2. Occupation time fluctuations limits

We give first the occupation time fluctuation limits for the two examples in the case of finite variance branching. In contrast with the general theorems (Theorems 2.2.1 – 2.2.3) we present the results in a different order, ending up with the classical $t^{1/2}$–norming in each case. This is because the emphasis now, in the α–stable case, is in going from the intermediate to the high dimensions d.

We denote by $\mathcal{N}$ a real–valued centered Gaussian random variable whose variance is specified in each case.

Theorem 3.2.1. Let W_t be either the α–stable process in $\mathbb{R}^d$ or the c–hierarchical random walk in Ω_N.

0–level:

(i) For $-1 < \gamma < 0, (t^{1-\gamma}\widetilde{h}_t^{(2,\gamma)})^{-1/2} Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$2q_\gamma^{(0)}.$$

(ii) For $\gamma = 0, (t \log t)^{-1/2} Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$2q_0^{(0)}.$$

(iii) For $\gamma > 0, t^{-1/2} Y_t$ converges to a Gaussian field with covariance kernel

$$2Q_{\gamma,1}(x).$$

1–level:

(i) For $0 < \gamma < 1, (t^{2-\gamma}\widetilde{h}_t^{(3,\gamma-1)})^{-1/2} Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$Vq_\gamma^{(1)}.$$

(ii) For $\gamma = 1, (t \log t)^{-1/2} Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$Vq_1^{(1)}.$$

(iii) For $\gamma > 1, t^{-1/2} Y_t$ converges to a Gaussian field with covariance kernel

$$2Q_{\gamma,1}(x) + VQ_{\gamma,2}(x).$$

2–level:

(i) For $1 < \gamma < 2$, $(t^{3-\gamma}\widetilde{h}_t^{(3,\gamma-2)})^{-1/2}Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$\frac{V_1V_2}{2}q_\gamma^{(2)}.$$

(ii) For $\gamma = 2$, $(t\log t)^{-1/2}Y_t$ converges to $\mathcal{N}\rho$, where $\mathcal{N}$ has variance

$$\frac{V_1V_2}{2}q_2^{(2)}.$$

(iii) For $\gamma > 2$, $t^{-1/2}Y_t$ converges to a Gaussian field with covariance kernel

$$2Q_{\gamma,1}(x) + (V_1+V_2)Q_{\gamma,2}(x) + \frac{V_1V_2}{2}Q_{\gamma,3}(x).$$

The correspondences for the examples are as follows (recall that a_t denotes the normalization):

For the α stable process: $\gamma = \dfrac{d}{\alpha} - 1$, in this case the functions $\widetilde{h}^{(\cdot,\cdot)}$ are constant, and the constants are included in the q_γ's.

0–level:

(i) $\alpha > d$, $a_t = t^{(1-d/2\alpha)}$, $q_\gamma^{(0)} = \dfrac{\kappa_{d,\gamma}}{(1-\gamma)(-\gamma)} = \dfrac{\kappa'_{d,\alpha}}{(2-d/\alpha)(1-d/\alpha)}$.

(ii) $\alpha = d, a_t = (t\log t)^{1/2}, q_0^{(0)} = \kappa_{d,0} = \kappa'_{d,d}$.

(iii) $\alpha < d, a_t = t^{1/2}, Q_{\gamma,1}(x) = G_{d,\gamma,1}(x) = G'_{d,\alpha,1}(x)$.

1–level:

(i) $\dfrac{d}{2} < \alpha < d$, $a_t = t^{(3/2-d/2\alpha)}$, $q_\gamma^{(1)} = \dfrac{\kappa_{d,\gamma}(2-2^{1-\gamma})}{(2-\gamma)(1-\gamma)\gamma} = \dfrac{\kappa'_{d,\alpha}(2-2^{2-d/\alpha})}{(3-d/\alpha)(2-d/\alpha)(d/\alpha-1)}$.

(ii) $\alpha = \dfrac{d}{2}, a_t = (t\log t)^{1/2}, q_1^{(1)} = \kappa_{d,1} = \kappa'_{d,d/2}$.

(iii) $\alpha < \dfrac{d}{2}, a_t = t^{1/2}, Q_{\gamma,j}(x) = G_{d,\gamma,j}(x) = G'_{d,\alpha,j}(x), j = 1,2$.

2–level:

(i) $\dfrac{d}{3} < \alpha < \dfrac{d}{2}$, $a_t = t^{(2-d/2\alpha)}$, $q_\gamma^{(2)} = \dfrac{\kappa_{d,\gamma}(2-2^{2-\gamma})}{(3-\gamma)(2-\gamma)(\gamma-1)\gamma} = \dfrac{\kappa'_{d,\alpha}(2-2^{3-d/\alpha})}{(4-d/\alpha)(3-d/\alpha)(d/\alpha-2)(d/\alpha-1)}$.

(ii) $\alpha = \dfrac{d}{3}, a_t = (t\log t)^{1/2}, q_2^{(2)} = \kappa_{d,2} = \kappa'_{d,d/3}$.

(iii) $\alpha < \dfrac{d}{3}, a_t = t^{1/2}, Q_{\gamma,j}(x) = G_{d,\gamma,j}(x) = G'_{d,\alpha,j}(x), j = 1,2,3.$

For the c–hierarchical random walk: $\gamma = \dfrac{\log c}{\log(N/c)}, c = N^{\gamma/(\gamma+1)}$.

0–level:

(i) $c < 1$, $a_t = t^{\log(N^{1/2}/c)/\log(N/c)}(\widetilde{h}_t^{(2,\gamma)})^{1/2}, q_\gamma^{(0)} = \kappa_{N,\gamma} = \kappa'_{N,c}$.

(ii) $c = 1, a_t = (t\log t)^{1/2}, q_0^{(0)} = \dfrac{\kappa_{N,0}}{-\log a} = \dfrac{(N-1)^2}{\sigma(N^2-1)\log N}$.

(iii) $c > 1, a_t = t^{1/2}, Q_{\gamma,1}(x) = G_{N,\gamma,1}(x) = G'_{N,c,1}(x)$.

1–level:

(i) $1 < c < N^{1/2}$, $a_t = t^{\log(N/c^{3/2})/\log(N/c)}(\widetilde{h}_t^{(3,\gamma-1)})^{1/2}, q_\gamma^{(1)} = \kappa_{N,\gamma} = \kappa'_{N,c}$.

(ii) $c = N^{1/2}, a_t = (t\log t)^{1/2}, q_1^{(1)} = \dfrac{\kappa_{N,1}}{\log a^{-1/2}} = \dfrac{2(N-1)^3}{(\sigma(N^{3/2}-1))^2\log N}$.

(iii) $c > N^{1/2}, Q_{\gamma,j}(x) = G_{N,\gamma,j}(x) = G'_{N,c,j}(x), j = 1,2.$

2–level:

(i) $N^{1/2} < c < N^{2/3}$, $a_t = t^{\log(N^{3/2}/c^2)/\log(N/c)}(\widetilde{h}_t^{(3,\gamma-2)})^{1/2}, q_\gamma^{(2)} = \kappa_{N,\gamma} = \kappa'_{N,c}$.

(ii) $c = N^{2/3}, a_t = (t\log t)^{1/2}, q_2^{(2)} = \dfrac{\kappa_{N,2}}{\log a^{-1/3}} = \dfrac{2(N-1)^4}{(\sigma(N^{4/3}-1))^3\log N}$.

(iii) $c > N^{2/3}, a_t = t^{1/2}, Q_{\gamma,j}(x) = G_{N,\gamma,j}(x) = G'_{N,c,j}(x), j = 1,2,3.$

We turn now to the case of infinite variance branching with α-stable motion in $\mathbb{R}^d$. The question is when do the assumptions for Theorems 2.5.1 and 2.5.2 hold.

Proposition 3.2.1. For the symmetric α–stable motion in $\mathbb{R}^d$, condition (2.5.1) holds if and only if $d > \alpha\left(1 + \dfrac{1}{\beta}\right)$.

Proposition 3.2.2. For the symmetric α–stable motion in $\mathbb{R}^d$, condition (2.5.2) holds if and only if $\beta_2 < \beta_1$ and $d > \alpha\left(1 + \dfrac{1}{\beta_2}\left(1 + \dfrac{1}{\beta_1}\right)\right)$.

3.3. Comments on the results

1. Note that for the 2–level branching Brownian system ($\alpha = 2$) the normings $t^{3/4}, (t \log t)^{1/2}$ and $t^{1/2}$ mentioned in the Introduction correspond to dimensions $d = 5, d = 6$ and $d > 6$, respectively, i.e., two dimensions higher than for the 1–level system. In the Brownian case the critical dimensions for the 1– and 2–level branching systems are $d = 4$ and $d = 6$, corresponding to $\gamma = 1$ and $\gamma = 2$, respectively. For 1–level Brownian systems, occupation time large deviation results have been obtained by Deuschel and Rosen[13] (and references therein).

2. For the α–stable process in $\mathbb{R}^d$ the order of transience/recurrence parameter γ defined in (3.1.1) can take values only in the interval $[(d-2)/2, \infty)$. On the other hand, for the c-hierarchical random walk in Ω_N the possible values of the parameter γ defined in (3.1.4) range over the whole interval $(-1, \infty)$, which means that in Ω_N there is a rich structure of naturally ordered random walks.

3. Note that for both the α–stable motion and the c–hierarchical random walk, in all cases (i) and (ii) of Theorem 3.2.1 the occupation time fluctuation limits in different regions of the space S are perfectly correlated. An intuitive explanation for this might come from the recurrent visits of each k–level equilibrium clan, $k = 0, 1, 2$, to all bounded regions $B \subset S$.

4. Equating the parameters γ for the α–stable process (3.1.1) and the c–hierarchical random walk (3.1.4) we obtain
$$c = N^{1-\alpha/d}.$$
For this value of c, by Lemma 3.1.1 the c–hierarchical random walk in Ω_N and the α–stable process in $\mathbb{R}^d$ have the same order of transience/recurrence. Consequently, by Theorems 2.2.1, 2.2.2 and 2.2.3 the asymptotics of the occupation time fluctuations are analogous for the corresponding k–level branching systems, $k = 0, 1, 2$. The only differences are in the constants and the kernels of the powers of the Green operators which appear in the fluctuation limits in Theorem 3.2.1. The same observation holds for branching systems of "α–stable" random walks on the lattice $\mathbb{Z}^d$. In passing we note that α–stable motions with $\alpha < 2$ do not have finite moments of order $\leq \alpha$, but this plays no role in the asymptotics of the occupation times. The corresponding c–hierarchical random walks have finite moments of all orders.

5. For the c–hierarchical random walk with $c = c_N = \eta N^{k/(k+1)}$, $k \geq 0$, $\eta > 1$, the powers of the Green potential operator take a simple form in the limit $N \to \infty$: all the powers of order $1 \leq j \leq k$ vanish as $N \to \infty$, and for the $(k+1)$–st power we observe from (4.2.2) and Lemma 3.1.2(b) that
$$\lim_{N\to\infty} G_{N,c_N,k+1}(x) = \frac{\eta^{k+1}}{\sigma^{k+1}(\eta^{k+1}-1)}(\eta^{k+1})^{-|x|}.$$

In particular, $\lim_{N\to\infty} G_{N,\eta,1}$ and $\lim_{N\to\infty} G_{N,\eta N^{1/2},2}$ have the same spatial asymptotics. This indicates that "near $\eta = 1$" a similar analysis as was carried out by Dawson and Greven(7) for 1--level branching hierarchical random walks ($k = 0$) might also be possible for 2–level branching hierarchical random walks ($k = 1$).

6. The results for the α–stable motion with the $t^{1/2}$–norming can be extended to test functions in $\mathcal{C}_\tau^+(\mathbb{R}^d)$. For example, for the 2–level system (with $d > 3\alpha$) we can take $\tau(x) = (1 + |x|^2)^{-q}$ with $d/2 < q < (d+\alpha)/2$ (Dawson and Gorostiza(6)). Moreover, the results can be extended to convergence of $\mathcal{S}'(\mathbb{R}^d)$–valued random fields, where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$, using an argument of Iscoe(27).

7. Similarly to the previous comment, for the c–hierarchical random walk the results with the $t^{1/2}$–norming can be extended to test functions in $\mathcal{C}_\tau^+(\Omega_N)$ with an appropriate function τ. For example, for the 2–level system, τ should be a function in $L^1(\Omega_N, \nu)$ such that the function $x \mapsto \sum\limits_y \tau(y)(N^2/c^3)^{|x-y|}$ is bounded.

4. DEFINITIONS OF CONSTANTS AND FUNCTIONS FOR THE EXAMPLES

4.1. Notation for α–stable motion:

$$\kappa_{d,\gamma} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|x|^{d/(\gamma+1)}}dx = \kappa'_{d,\alpha} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|x|^\alpha}dx, \tag{4.1.1}$$

$$G_{d,\gamma,j}(x) = C_{d,\gamma,j}|x|^{-d(1-j/(\gamma+1))} = G'_{d,\alpha,j}(x) = C'_{d,\alpha,j}|x|^{-(d-j\alpha)}, \tag{4.1.2}$$

where

$$C_{d,\gamma,j} = \frac{\Gamma\left(\dfrac{d(\gamma+1-j)}{2(\gamma+1)}\right)}{2^{jd/(\gamma+1)}\pi^{d/2}\Gamma\left(\dfrac{dj}{2(\gamma+1)}\right)} = C'_{d,\alpha,j} = \frac{\Gamma\left(\dfrac{d-j\alpha}{2}\right)}{2^{j\alpha}\pi^{d/2}\Gamma\left(\dfrac{j\alpha}{2}\right)},$$

and j is a positive integer such that $j < \gamma + 1$, i.e. $\alpha j < d$.

4.2. Notation for c–hierarchical random walk:

$$\kappa_{N,\gamma} = (N-1)^{\gamma+2}(\sigma(N^{(\gamma+2)/(\gamma+1)} - 1))^{-(\gamma+1)} \tag{4.2.1}$$

$$= \kappa'_{N,c} = (N-1)^{\log(N^2/c)/\log(N/c)}(\sigma(N^2/c-1))^{-\log N/\log(N/c)},$$

$$G_{N,\gamma,j}(x) = C_{N,\gamma,j} N^{-|x|(1-j/(\gamma+1))} \;=\; G'_{N,c,j}(x) = C'_{N,c,j} \left(\frac{N^{j-1}}{c^j} \right)^{|x|}, \tag{4.2.2}$$

where

$$\begin{aligned} C_{N,\gamma,j} &= \left(\frac{N-1}{\sigma(N^{(\gamma+2)/(\gamma+1)}-1)} \right)^j \left[\delta_{0,|x|} - 1 + \frac{(N-1)N^{j-1}}{N^{j\gamma/(\gamma+1)} - N^{j-1}} \right] \\ = \quad C'_{N,\gamma,j} &= \left(\frac{N-1}{\sigma(N^2/c-1)} \right)^j \left[\delta_{0,|x|} - 1 + \frac{(N-1)N^{j-1}}{c^j - N^{j-1}} \right], \end{aligned}$$

and j is a positive integer such that $j < \gamma + 1$, i.e. $c > N^{(j-1)/j}$.

Note that a and b defined in (3.1.2) are also expressed as

$$a = N^{-1/(\gamma+1)}, \quad b = \frac{N^{(\gamma+2)/(\gamma+1)} - 1}{N-1}. \tag{4.2.3}$$

To simplify notation we write

$$\theta = \sigma \frac{N^{(\gamma+2)/(\gamma+1)} - 1}{N-1}. \tag{4.2.4}$$

For $\zeta > 0$, let

$$h_t^{(1,\zeta)} = \sum_{j=-\infty}^{\infty} (\theta a^j t)^\zeta e^{-\theta a^j t}, \qquad t > 0, \tag{4.2.5}$$

and for $-1 < \zeta < 0$, let

$$h_t^{(2,\zeta)} = \sum_{j=-\infty}^{\infty} (\theta a^j t)^\zeta (1 - e^{-\theta a^j t}), \qquad t > 0, \tag{4.2.6}$$

$$h_t^{(3,\zeta)} = \sum_{j=-\infty}^{\infty} (\theta a^j t)^\zeta (1 - e^{-\theta a^j t})^2, \qquad t > 0, \tag{4.2.7}$$

$$\widetilde{h}_t^{(2,\zeta)} = \sum_{j=-\infty}^{\infty} (\theta a^j t)^{\zeta-1} (e^{-\theta a^j t} - 1 + \theta a^j t), \qquad t > 0, \tag{4.2.8}$$

$$\widetilde{h}_t^{(3,\zeta)} = \sum_{j=-\infty}^{\infty} (\theta a^j t)^{\zeta-1} \left(2e^{-\theta a^j t} - \frac{1}{2} e^{-2\theta a^j t} + \theta a^j t - \frac{3}{2} \right), \qquad t > 0, \tag{4.2.9}$$

with a and θ given by (4.2.3) and (4.2.4). Note that all the functions defined in (4.2.5)–(4.2.9) belong to $\widetilde{\mathcal{H}}_a$ for a given by (4.2.3). The functions $\widetilde{h}^{(2,\zeta)}$ and $\widetilde{h}^{(3,\zeta)}$ correspond (asymptotically) to $h^{(2,\zeta)}$ and $h^{(3,\zeta)}$, respectively, by Lemma 2.4.3, but in this case they are obtained by explicit calculation of the l.h.s. of (2.4.1).

5. PROOFS

5.1. Asymptotics of the powers of G_t

m *Proof of Lemma 2.4.2:*

Note that it suffices to do the proofs with $\varphi = \psi$. Fix $\varphi \in \mathcal{C}_c^+(S), \varphi \neq 0$, and denote $J = J(\varphi, \varphi)$ and $R_t = R_t(\varphi, \varphi)$. By assumption, $R_t \sim t^{-\gamma} J$.

(1)
$$(\varphi, G_t G\varphi)_\rho = \left(\varphi, \int_0^t \int_0^\infty T_{s+u}\varphi ds du\right)_\rho = \int_0^t R_u du \sim J\frac{1}{1-\gamma}t^{1-\gamma},$$

and

$$(\varphi, (G_t G - G_t^2)\varphi)_\rho = \left(\varphi, G_t \int_t^\infty T_s \varphi ds\right)_\rho = \left(\varphi, \int_0^t \int_t^\infty T_{s+u}\varphi ds du\right)_\rho = \int_0^t R_{s+t} ds$$
$$\sim J\int_0^t (s+t)^{-\gamma} ds \;=\; J\frac{1}{1-\gamma}((2t)^{1-\gamma} - t^{1-\gamma}) \;=\; \frac{J}{1-\gamma}(2^{1-\gamma}-1)t^{1-\gamma},$$

hence

$$(\varphi, G_t^2\varphi)_\rho = (\varphi, (G_t^2 - G_t G)\varphi)_\rho + (\varphi, G_t G\varphi)_\rho \sim \frac{J}{1-\gamma}(2-2^{1-\gamma})t^{1-\gamma}.$$

(2)
$$(\varphi, G_t G\varphi)_\rho = \int_0^t R_u du \sim J\int_1^t u^{-1} du \sim c\log t,$$
$$(\varphi, (G_t G - G_t^2)\varphi)_\rho = \int_0^t R_{s+t} ds$$
$$\sim J\int_0^t (s+t)^{-1} ds \sim J(\log(2t) - \log t) = o(\log t),$$

hence the assertion for G_t^2 follows.

(3)
$$(\varphi, G_t^2 G\varphi)_\rho = \int_0^t \int_0^t R_{s+u} dr du$$
$$\sim J\int_1^t \int_1^t (s+u)^{-\gamma} ds du \sim \frac{J}{(2-\gamma)(\gamma-1)}(2-2^{2-\gamma})t^{2-\gamma},$$
$$(\varphi, (G_t^2 G - G_t^3)\varphi)_\rho = \int_0^t \int_0^t R_{s+u+t} du ds$$
$$\sim J\int_0^t \int_0^t (s+u+t)^{-\gamma} du ds \sim \frac{J}{(2-\gamma)(\gamma-1)}(-3^{2-\gamma} + 2\cdot 2^{2-\gamma} - 1)t^{2-\gamma},$$

hence the assertion for G_t^3 follows.

(4)
$$(\varphi, G_t^2 G\varphi)_\rho = \int_0^t \int_0^t R_{s+u} dr du$$
$$\sim J \int_1^t \int_0^t (s+u)^{-2} ds du \sim c \log t,$$
$$(\varphi, (G_t^2 G - G_t^3)\varphi)_\rho = \int_0^t \int_0^t R_{s+u+t} du ds$$
$$\sim J \int_0^t \int_0^t (s+u+t)^{-2} ds du = o(\log t),$$

hence the assertion for G_t^3 follows. □

Proof of Lemma 2.4.3:
$$\int_1^t s^\zeta h_s ds = -\log a \cdot \int_0^\tau a^{-r(1+\zeta)} h_{a^{-r}} dr,$$
where $\tau = -\dfrac{\log t}{\log a}$. Hence, since $a < 1$ and h is bounded, we have
$$\begin{aligned} t^{-(1+\zeta)} \int_1^t s^\zeta h_s ds &= -\log a \cdot \int_0^\tau a^{(\tau-r)(1+\zeta)} h_{a^{-r}} dr \\ &= -\log a \cdot \int_0^\tau a^{r(1+\zeta)} h_{a^{-(\tau-r)}} dr \\ &\sim -\log a \cdot \int_0^\infty a^{r(1+\zeta)} h_{a^r t} dr. \end{aligned}$$
□

5.2. Main results

We will not include the proofs for the 0–level system (Theorem 2.2.1) because they are simpler versions of the proofs for the branching systems. The proofs for the 1– and 2–level systems follow the idea of the method employed by Iscoe[27] for (1–level) superprocesses in $\mathbb{R}^d$. The particle systems are somewhat harder to deal with than the superprocesses, but the main point is it has been necessary to modify the method in order to deal with the new technical difficulties that arise from the second level branching. We will use the modified approach also for the 1–level system.

Proof of Theorems 2.2.2 and 2.5.1 (1–level branching system):

In order to simplify notation we write
$$\varphi_t = F_t^{-1/(1+\beta)} \varphi \quad \text{for } \varphi \in \mathcal{C}_c^+(S), \varphi \neq 0 \text{ and } \beta \leq 1, \text{ where } F_t = \int_0^t f_s ds, \tag{5.2.1}$$
and f_s is a growth function. For Theorem 2.2.2(a) and Theorem 2.5.1, f_t is interpreted as $f_t \equiv 1$.

We have, by (A.1.1) and (A.1.2) (Appendix),

$$E\exp\left\{-F_t^{-1/(1+\beta)}\left\langle\int_0^t X_s ds,\varphi\right\rangle\right\} = \exp\{-\langle\rho, u_{\varphi_t}(t)\rangle\},$$

where $u_{\varphi_t}(s,x)$ with values in $[0,1]$ is the unique solution of

$$u_{\varphi_t}(s) = -\frac{V}{1+\beta}\int_0^s T_{s-r}(u_{\varphi_t}(r)^{1+\beta})dr + \int_0^s T_{s-r}(\varphi_t(1-u_{\varphi_t}(r)))dr \tag{5.2.2}$$

Hence, by T_t–invariance of ρ and $E\langle\int_0^t X_s ds,\varphi\rangle = t\langle\rho,\varphi\rangle$, for the occupation time fluctuation Y_t we have

$$E\exp\{-F_t^{-1/(1+\beta)}\langle Y_t,\varphi\rangle\} = \exp\left\{\frac{V}{1+\beta}(I_1(t)+I_2(t)) + I_3(t))\right\}, \tag{5.2.3}$$

where

$$I_1(t) = \int_0^t \langle\rho, w_{\varphi_t}(s)^{1+\beta}\rangle ds, \tag{5.2.4}$$

$$I_2(t) = \int_0^t \langle\rho, u_{\varphi_t}(s)^{1+\beta} - w_{\varphi_t}(s)^{1+\beta}\rangle ds, \tag{5.2.5}$$

$$I_3(t) = \int_0^t \langle\rho, \varphi_t u_{\varphi_t}(s)\rangle ds, \tag{5.2.6}$$

with

$$w_\varphi(s)(x) := \int_0^s T_r\varphi(x)dr = G_s\varphi(x),\ \ x\in S, \varphi\in\mathcal{C}_c^+(S). \tag{5.2.7}$$

We will prove the following limits as $t\to\infty$:

For Theorem 2.2.2(a):

$$I_1(t)\to(\varphi, G^2\varphi)_\rho. \tag{5.2.8}$$

For Theorem 2.2.2(b):

$$I_1(t)\to H(\varphi,\varphi). \tag{5.2.9}$$

For Theorem 2.5.1:

$$I_1(t)\to\langle\rho,(G\varphi)^{1+\beta}\rangle. \tag{5.2.10}$$

For $\beta\le 1$:

$$I_2(t)\to 0. \tag{5.2.11}$$

For $\beta<1$:

$$I_3(t) \to 0. \tag{5.2.12}$$

For Theorem 2.2.2(a):

$$I_3(t) \to (\varphi, G\varphi)_\rho. \tag{5.2.13}$$

For Theorem 2.2.2(b):

$$I_3(t) \to 0. \tag{5.2.14}$$

These limits will yield the conclusions of the theorems.

Proof of (5.2.8) and (5.2.9): From (5.2.4) and (5.2.7) we have

$$I_1(t) = F_t^{-1} \int_0^t \langle \rho, (G_s\varphi)^2 \rangle ds.$$

By L'Hôpital's rule, for (5.2.8) we have

$$I_1(t) = t^{-1} \int_0^t \langle \rho, (G_s\varphi)^2 \rangle ds \sim (\varphi, G_t^2\varphi)_\rho \to (\varphi, G^2\varphi)_\rho,$$

and for (5.2.9),

$$I_1(t) = F_t^{-1} \int_0^t \langle \rho, (G_s\varphi)^2 \rangle ds \sim \frac{1}{f_t}(\varphi, G_t^2\varphi)_\rho \to H(\varphi, \varphi).$$

Proof of (5.2.10): The same as (5.2.8).

Proof of (5.2.11): We rewrite (5.2.5) as

$$-I_2(t) = \int_0^t \langle \rho, w_{\varphi_t}(s)^{1+\beta}[1 - (u_{\varphi_t}(s)/w_{\varphi_t}(s))^{1+\beta}] \rangle ds.$$

We have from (5.2.2) and (5.2.7)

$$u_{\varphi_t}(s) - w_{\varphi_t}(s) = -\frac{V}{1+\beta} \int_0^s T_{s-r}(u_{\varphi_t}(r)^{1+\beta}) ds - \int_0^s T_{s-r}(\varphi_t u_{\varphi_t}(r)) dr \leq 0, \tag{5.2.15}$$

hence

$$0 \leq 1 - (u_{\varphi_t}(s)/w_{\varphi_t}(s))^{1+\beta} \leq 1.$$

Therefore, since the convergence of $I_1(t)$ implies uniform integrability, it suffices to show that

$$\lim_{t\to\infty} \frac{u_{\varphi_t}(s)}{w_{\varphi_t}(s)} = 1 \ \text{ for all } s, \rho - a.e.$$

We have from (5.2.7) and (5.2.15)

$$\frac{u_{\varphi_t}(s)}{w_{\varphi_t}(s)} = 1 - (G_s\varphi)^{-1}(J_t(s) + K_t(s)),$$

where

$$J_t(s) = \frac{V}{1+\beta} F_t^{1/(1+\beta)} \int_0^s T_{s-r}(u_{\varphi_t}(r)^{1+\beta}) dr \geq 0,$$

and

$$K_t(s) = \int_0^s T_{s-r}(\varphi u_{\varphi_t}(r)) dr \geq 0.$$

By (5.2.15),

$$\begin{aligned} J_t(s) &\leq \frac{V}{1+\beta} F_t^{1/(1+\beta)} \int_0^s T_{s-r}(w_{\varphi_t}(r)^{1+\beta}) dr \\ &= \frac{V}{1+\beta} F_t^{-\beta/(1+\beta)} \int_0^s T_{r-s}((G_r\varphi)^{1+\beta}) dr \to 0. \end{aligned}$$

Similarly, $K_t(s) \to 0$, so (5.2.11) is proved.

Proof of (5.2.12): We have from (5.2.6) and (5.2.15)

$$I_3(t) \leq t^{-2/(1+\beta)} \int_0^t (\varphi, G_s\varphi)_\rho ds = t^{(\beta-1)/(1+\beta)} t^{-1} \int_0^t (\varphi, G_s\varphi)_\rho ds,$$

and the result follows since $(\varphi, G_s\varphi)_\rho \to (\varphi, G\varphi)_\rho < \infty$ as $s \to \infty$.

Proof of (5.2.13): We rewrite (5.2.6) as

$$I_3(t) = t^{-1} \int_0^t \langle \rho, \varphi \widetilde{u}_{\varphi_t}(s) \rangle ds,$$

where $\widetilde{u}_{\varphi_t}(s) := t^{1/2} u_{\varphi_t}(s)$. Since

$$t^{-1} \int_0^t \langle \rho, \varphi G_s \varphi \rangle ds \to (\varphi, G\varphi)_\rho,$$

it suffices to prove that

$$A_t := t^{-1} \int_0^t \langle \rho, \varphi \widetilde{u}_{\varphi_t}(s) \rangle ds - t^{-1} \int_0^t \langle \rho, \varphi G_s \varphi \rangle ds \to 0$$

We have, from (5.2.2)

$$\widetilde{u}_{\varphi_t}(s) = -\frac{V}{2} t^{-1/2} \int_0^s T_{s-r}(\widetilde{u}_{\varphi_t}(r)^2) dr + G_s\varphi - t^{-1/2} \int_0^s T_{s-r}(\varphi \widetilde{u}_{\varphi_t}(r)) dr,$$

hence

$$|A_t| \leq t^{-3/2} \left(\frac{V}{2} \int_0^t \int_0^s \langle \rho, \varphi T_{s-r}(\widetilde{u}_{\varphi_t}(r)^2) \rangle dr ds + \int_0^t \int_0^s \langle \rho, \varphi T_{s-r}(\varphi \widetilde{u}_{\varphi_t}(r)) \rangle dr ds \right),$$

but, from (5.2.2), $\widetilde{u}_{\varphi_t}(r) \leq G_r\varphi$ (since $f_t \equiv 1$), so

$$|A_t| \quad \leq \quad t^{-3/2} \left(\frac{V}{2} \int_0^t \int_0^s \langle \rho, \varphi T_{s-r}(G_r\varphi)^2 \rangle dr ds + \int_0^t \int_0^s \langle \rho, \varphi T_{s-r}(\varphi G_r \varphi) \rangle dr ds \right)$$

$$= t^{-3/2}\left(\frac{V}{2}\int_0^t \langle \rho, \varphi G_{t-r}(G_r\varphi)^2\rangle dr + \int_0^t \langle \rho, \varphi G_{t-r}(\varphi G_r\varphi)\rangle dr\right), \tag{5.2.16}$$

therefore

$$|A_t| \leq t^{-1/2}\left\langle \rho, \frac{V}{2}\varphi G(G\varphi)^2 + \varphi G(\varphi G\varphi)\right\rangle.$$

Now, by strong transience,

$$\langle \rho, \varphi G(G\varphi)^2\rangle \;\; \leq \;\; ||G\varphi|| \; ||G^2\varphi|| \langle \rho, \varphi\rangle < \infty,$$

and by transience,

$$\langle \rho, \varphi G(\varphi G\varphi)\rangle \;\; \leq \|G\varphi\|^2 \; \langle \rho, \varphi\rangle < \infty,$$

hence the result follows.

Proof of (5.2.14): We can follow the same argument used for (5.2.13) replacing φ by $\varphi F_t^{-1/2}$. Both terms on the r.h.s. of (5.2.16) can be shown to converge to 0 by L'Hôpital's rule and the assumptions.

It follows from (5.2.3)–(5.2.6) and the limits (5.2.8)–(5.2.14) that

$$E\exp\{-(F_t)^{-1/(1+\beta)}\langle Y_t, \varphi\rangle\}$$

$$\rightarrow \begin{cases} \exp\left\{\dfrac{V}{2}(\varphi, G^2\varphi)_\rho + (\varphi, G\varphi)_\rho\right\} & \text{for Theorem 2.2.2(a),} \\ \exp\left\{\dfrac{V}{2}H(\varphi,\varphi)\right\} & \text{for Theorem 2.2.2(b),} \\ \exp\left\{\dfrac{V}{1+\beta}\langle \rho, (G\varphi)^{1+\beta}\rangle\right\} & \text{for Theorem 2.5.1} \end{cases}$$

as $t \rightarrow \infty$.

Finally, the convergence of the bilateral Laplace functional implies the weak convergence of Y_t as $t \rightarrow \infty$ (Iscoe[27], pp. 106–107 and 112). □

Proof of Theorems 2.2.3 and 2.5.3 (2–level branching system):

We will follow the same steps for the proof of the 1–level case, but now some of them are harder. The problem is that, while the test functions $\varphi \in \mathcal{C}_c^+(S)$ and the measure ρ for the 1–level system are not so difficult to work with, for the 2–level system the test functions $\mu \mapsto \langle \mu, \varphi\rangle$ and the measure R_∞^1 (which now plays the role of ρ) raise new technical questions that are not easy to deal with, in particular involving the third moments of R_∞^1. The background on the 2–level system in the Appendix should be consulted at this point.

We continue to use the notation φ_t introduced in (5.2.1), but now with $\beta = \beta_2$, and we put $f_t \equiv 1$ for Theorem 2.2.3(a) and Theorem 2.5.2.

For $\varphi \in \mathcal{C}_c^+(S)$ we have, by (A.1.13) and (A.1.14) (Appendix),

$$E \exp\left\{-F_t^{-1/(1+\beta_2)}\left\langle \int_0^t X_s ds, \varphi\right\rangle\right\} = \exp\{-\langle\!\langle R_\infty^1, \mathbf{u}_{\varphi_t}(t)\rangle\!\rangle\},$$

where

$$\mathbf{u}_{\varphi_t}(s) = -\frac{V_2}{1+\beta_2}\int_0^s U_{s-r}(\mathbf{u}_{\varphi_t}(r)^{1+\beta_2})dr + \int_0^s U_{s-r}(\langle\cdot, \varphi_t\rangle(1-\mathbf{u}_{\varphi_t}(r)(\cdot)))dr. \tag{5.2.17}$$

Hence, by U_t–invariance of R_∞^1 and $E\langle\int_0^t X_s ds, \varphi\rangle = t\langle\rho, \varphi\rangle$,

$$E\exp\{-F_t^{-1/(1+\beta_2)}\langle Y_t, \varphi\rangle\} = \exp\left\{\frac{V_2}{1+\beta_2}(I_1(t) + I_2(t)) + I_3(t))\right\}, \tag{5.2.18}$$

where

$$I_1(t) = \int_0^t \langle\!\langle R_\infty^1, \mathbf{w}_{\varphi_t}(s)^{1+\beta_2}\rangle\!\rangle ds, \tag{5.2.19}$$

$$I_2(t) = \int_0^t \langle\!\langle R_\infty^1, \mathbf{u}_{\varphi_t}(s)^{1+\beta_2} - \mathbf{w}_{\varphi_t}(s)^{1+\beta_2}\rangle\!\rangle ds, \tag{5.2.20}$$

$$I_3(t) = \int_0^t \langle\!\langle R_\infty^1, \langle\cdot, \varphi_t\rangle \mathbf{u}_{\varphi_t}(s)(\cdot)\rangle\!\rangle ds, \tag{5.2.21}$$

with, by (A.1.4),

$$\mathbf{w}_\varphi(s)(\mu) := \int_0^s U_r(\langle\cdot, \varphi\rangle)(\mu)dr = \int_0^s \langle\mu, T_r\varphi\rangle dr = \langle\mu, G_s\varphi\rangle, \quad \mu \in \mathcal{M}_\tau(S). \tag{5.2.22}$$

We will prove the following limits as $t \to \infty$:

For Theorem 2.2.3(a):

$$I_1(t) \to (\varphi, G^2\varphi)_\rho + \tfrac{V_1}{2}(\varphi, G^3\varphi)_\rho. \tag{5.2.23}$$

For Theorem 2.2.3(b):

$$I_1(t) \to \frac{V_1}{2}H(\varphi, \varphi). \tag{5.2.24}$$

For Theorem 2.5.2:

$$I_1(t) \to \langle\!\langle R_\infty^1, \langle\cdot, G\varphi\rangle^{1+\beta_2}\rangle\!\rangle. \tag{5.2.25}$$

For $\beta_2 \leq 1$:

$$I_2(t) \to 0. \tag{5.2.26}$$

For $\beta_2 < 1$:

$$I_3(t) \to 0. \tag{5.2.27}$$

For Theorem 2.2.3(a):

$$I_3(t) \to (\varphi, G\varphi)_\rho + \frac{V_1}{2}(\varphi, G^2\varphi)_\rho. \tag{5.2.28}$$

For Theorem 2.2.3(b):

$$I_3(t) \to 0. \tag{5.2.29}$$

Proof of (5.2.23) and (5.2.24): We have from (5.2.19) and (5.2.22)

$$I_1(t) = F_t^{-1} \int_0^t \langle\langle R_\infty^1, \langle \cdot, G_s\varphi\rangle^2 \rangle\rangle ds.$$

By L'Hôpital's rule and using (A.1.12) we have, for (5.2.23),

$$\begin{aligned} I_1(t) &= t^{-1} \int_0^t \langle\langle R_\infty^1, \langle \cdot, G_s\varphi\rangle^2 \rangle\rangle ds \sim \langle\langle R_\infty^1, \langle \cdot, G_t\varphi\rangle^2 \rangle\rangle \\ &= (\varphi, G_t^2\varphi)_\rho + \frac{V_1}{2}(\varphi, G_t^2 G\varphi)_\rho \to (\varphi, G^2\varphi)_\rho + \frac{V_1}{2}(\varphi, G^3\varphi)_\rho, \end{aligned}$$

and for (5.2.24),

$$\begin{aligned} I_1(t) &= F_t^{-1} \int_0^t \langle\langle R_\infty^1, \langle \cdot, G_s\varphi\rangle^2 \rangle\rangle ds \\ &\sim \frac{1}{f_t} \langle\langle R_\infty^1, \langle \cdot, G_t\varphi\rangle^2 \rangle\rangle \\ &= \frac{1}{f_t}\left((\varphi, G_t^2\varphi)_\rho + \frac{V_1}{2}(\varphi, G_t^2 G\varphi)_\rho \right) \\ &\to \frac{V_1}{2} H(\varphi, \varphi). \end{aligned}$$

Proof of (5.2.25): The same as (5.2.23).

Proof of (5.2.26): We rewrite (5.2.20) as

$$\begin{aligned} -I_2(t) &= \int_0^t \langle\langle R_\infty^1, \mathbf{w}_{\varphi_t}(s)^{1+\beta_2} - \mathbf{u}_{\varphi_t}(s)^{1+\beta_2} \rangle\rangle ds \\ &= \int_0^t \langle\langle R_\infty^1, \mathbf{w}_{\varphi_t}(s)^{1+\beta}[1 - (\mathbf{u}_{\varphi_t}(s)/\mathbf{w}_{\varphi_t}(s))^{1+\beta_2}] \rangle\rangle ds. \end{aligned}$$

Since $0 \le 1 - (\mathbf{u}_{\varphi_t}(s)/\mathbf{w}_{\varphi_t}(s))^{1+\beta_2} \le 1$ by (5.2.2) and (A.1.16), and since $I_1(t)$ converges, it suffices to prove that

$$\lim_{t\to\infty} \frac{\mathbf{u}_{\varphi_t}(s)}{\mathbf{w}_{\varphi_t}(s)} = 1 \text{ for all } s, R_\infty^1 - a.e.$$

We have from (5.2.22) and (A.1.16)

$$\frac{\mathbf{u}_{\varphi_t}(s)(\mu)}{\mathbf{w}_{\varphi_t}(s)(\mu)} = 1 - \langle \mu, G_s\varphi\rangle^{-1}(J_t(s) + K_t(s))(\mu) \geq 0,$$

where

$$J_t(s)(\mu) = \frac{V_2}{1+\beta_2} F_t^{1/(1+\beta_2)} \int_0^s U_{s-r}(\mathbf{u}_{\varphi_t}(r)^{1+\beta_2})(\mu)dr \geq 0$$

and

$$K_t(s)(\mu) = \int_0^s U_{s-r}(\langle \cdot, \varphi\rangle \mathbf{u}_{\varphi_t}(r)(\cdot))(\mu)dr \geq 0.$$

By (A.1.17) and (5.2.22)

$$\begin{aligned} J_t(s)(\mu) &\leq \frac{V_2}{1+\beta_2} F_t^{1/(1+\beta_2)} \int_0^s U_{s-r}(\mathbf{w}_{\varphi_t}(r)^{1+\beta_2})(\mu)dr \\ &= \frac{V_2}{1+\beta_2} F_t^{-\beta_2/(1+\beta_2)} \int_0^s U_{s-r}(\langle \cdot, G_r\varphi\rangle^{1+\beta_2})(\mu)dr \to 0. \end{aligned}$$

Similarly, $K_t(s)(\mu) \to 0$, so the result follows.

Proof of (5.2.27): We have from (5.2.21), (5.2.22) and (A.1.12)

$$\begin{aligned} I_3(t) &\leq t^{-2/(1+\beta_2)} \int_0^t \langle\langle R^1_\infty, \langle \cdot, \varphi\rangle\langle \cdot, G_s\varphi\rangle\rangle\rangle ds \\ &= t^{(\beta_2-1)/(1+\beta_2)} t^{-1} \int_0^t \left((\varphi, G_s\varphi)_\rho + \frac{V_1}{2}(\varphi, G_sG\varphi)_\rho \right) ds. \end{aligned}$$

Since

$$(\varphi, G_s\varphi)_\rho + \frac{V_1}{2}(\varphi, G_sG\varphi) \to (\varphi, G\varphi)_\rho + \frac{V_1}{2}(\varphi, G^2\varphi)_\rho \quad \text{as } s \to \infty,$$

the result follows.

Proof of (5.2.28): We rewrite (5.2.21) as

$$I_3(t) = t^{-1} \int_0^t \langle\langle R^1_\infty, \langle \cdot, \varphi\rangle \widetilde{\mathbf{u}}_{\varphi_t}(s)(\cdot)\rangle\rangle ds,$$

where $\widetilde{\mathbf{u}}_{\varphi_t}(s) = t^{1/2}\mathbf{u}_{\varphi_t}(s)$. Since

$$t^{-1} \int_0^t \langle\langle R^1_\infty, \langle \cdot, \varphi\rangle\langle \cdot, G_s\varphi\rangle\rangle\rangle \;\to\; (\varphi, G\varphi)_\rho + \frac{V_1}{2}(\varphi, G^2\varphi)_\rho,$$

by (A.1.12), it suffices to prove that

$$A_t := t^{-1} \int_0^t \langle\langle R^1_\infty, \langle \cdot, \varphi\rangle \widetilde{\mathbf{u}}_{\varphi_t}(s)(\cdot)\rangle\rangle ds - t^{-1} \int_0^t \langle\langle R^1_\infty, \langle \cdot, \varphi\rangle\langle \cdot, G_s\varphi\rangle\rangle\rangle ds \to 0.$$

We have from (A.1.16)

$$\widetilde{\mathbf{u}}_{\varphi_t}(s)(\mu) = -\frac{V_2}{2} t^{-1/2} \int_0^s U_{s-r}(\widetilde{\mathbf{u}}_{\varphi_t}(r)^2)(\mu)dr + \langle \mu, G_s\varphi\rangle - t^{-1/2} \int_0^s U_{s-r}(\langle \cdot, \varphi\rangle \widetilde{\mathbf{u}}_{\varphi_t}(r)(\cdot))(\mu)dr,$$

hence

$$
\begin{aligned}
|A_t| &\le t^{-3/2}\left(\frac{V_2}{2}\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle U_{s-r}(\widetilde{\mathbf{u}}_{\varphi_t}(r)^2)(\mu)drds\right.\\
&\qquad\left.+\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle U_{s-r}(\langle\cdot,\varphi\rangle\widetilde{\mathbf{u}}_{\varphi_t}(r))(\mu)drds\right).
\end{aligned}
$$

Now $\widetilde{\mathbf{u}}_{\varphi_t}(r)(\mu)\le\langle\mu,G_r\varphi\rangle$ (since $f_t\equiv 1$). Note that this estimate is not too rough because $\widetilde{\mathbf{u}}_{\varphi_t}(r)(\mu)\to\langle\mu,G_r\varphi\rangle$ as $t\to\infty$. Hence

$$
|A_t|\le const.(H_1(t)+H_2(t)),
$$

where

$$
\begin{aligned}
H_1(t) &= t^{-3/2}\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle U_{s-r}(\langle\cdot,G_r\varphi\rangle^2)(\mu)drds,\\
H_2(t) &= t^{-3/2}\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle U_{s-r}(\langle\cdot,\varphi\rangle\langle\cdot,G_r\varphi\rangle)(\mu)drds.
\end{aligned}
$$

We will show that $H_1(t)\to 0$. The proof that $H_2(t)\to 0$ is similar.

Using (A.1.8) we have

$$
H_1(t)\le\ const.\sum_{j=1}^3 J_j(t),
$$

where

$$
\begin{aligned}
J_1(t) &= t^{-3/2}\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle\langle\mu,T_{s-r}G_r\varphi\rangle^2 drds,\\
J_2(t) &= t^{-3/2}\int_0^t\int_0^s\int R^1_\infty(d\mu)\langle\mu,\varphi\rangle\langle\mu,T_{s-r}(G_r\varphi)^2\rangle drds,\\
J_3(t) &= t^{-3/2}\int_0^t\int_0^s\int R^1_\infty(\mu)\langle\mu,\varphi\rangle\int_0^{s-r}\langle\mu,T_u(T_{s-r-u}G_r\varphi)^2\rangle dudrds.
\end{aligned}
$$

By (A.1.12) and (A.1.13) we obtain

$$
J_1(t)\le const.\sum_{j=1}^5 K_{1,j}(t),\quad J_2(t)\le const.\sum_{j=1}^2 K_{2,j}(t),\quad J_3(t)\le const.\sum_{j=1}^2 K_{3,j}(t),
$$

where

$$
\begin{aligned}
K_{1,1}(t) &= t^{-3/2}\int_0^t\int_0^s\langle\rho,\varphi(T_{s-r}G_r\varphi)^2\rangle drds,\\
K_{1,2}(t) &= t^{-3/2}\int_0^t\int_0^s\langle\rho,\varphi T_{s-r}G_r\varphi\cdot T_{s-r}GG_r\varphi\rangle drds,\\
K_{1,3}(t) &= t^{-3/2}\int_0^t\int_0^s\langle\rho,\varphi G(T_{s-r}G_r\varphi)^2\rangle drds,\\
K_{1,4}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^\infty\left\langle\rho,\varphi GT_u(T_uT_{s-r}G_r\varphi)^2\right\rangle dudrds,
\end{aligned}
$$

$$
\begin{aligned}
K_{1,5}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^\infty \langle\rho, T_{s-r}G_r\varphi\cdot GT_u(T_u\varphi\cdot T_uT_{s-r}G_r\varphi)\rangle dudrds,\\
K_{2,1}(t) &= t^{-3/2}\int_0^t\int_0^s \langle\rho, \varphi T_{s-r}(G_r\varphi)^2\rangle drds,\\
K_{2,2}(t) &= t^{-3/2}\int_0^t\int_0^s \langle\rho, \varphi GT_{s-r}(G_r\varphi)^2\rangle drds,\\
K_{3,1}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^{s-r} \left\langle\rho, \varphi T_u(T_{s-r-u}G_r\varphi)^2\right\rangle dudrds,\\
K_{3,2}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^{s-r} \langle\rho, \varphi GT_u(T_{s-r-u}G_r\varphi)^2\rangle dudrds.
\end{aligned}
$$

We will show that each of these terms converges to 0 as $t\to\infty$. Recall that $||G^j\varphi||<\infty, j=1,2,3$, for $\varphi\in\mathcal{C}_c^+(S)$.

$$K_{1,1}(t)\le ||G\varphi||t^{-3/2}\int_0^t\langle\rho,\varphi G_sG\varphi\rangle ds\le ||G\varphi||\ ||G^2\varphi||\langle\rho,\varphi\rangle t^{-1/2}\to 0.$$

$$K_{1,2}(t)\le ||G^2\varphi||t^{-3/2}\int_0^t\langle\rho,\varphi G_sG\varphi\rangle ds\le ||G^2\varphi||^2\langle\rho,\varphi\rangle t^{-1/2}\to 0.$$

$$K_{1,3}(t)\le ||G\varphi||t^{-3/2}\int_0^t\langle\rho,\varphi G_sG^2\varphi\rangle ds\le ||G\varphi||\ ||G^3\varphi||\langle\rho,\varphi\rangle t^{-1/2}\to 0.$$

$$
\begin{aligned}
K_{1,4}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^\infty\langle\rho,\varphi GT_u(T_uT_rG_{s-r}\varphi)^2\rangle dudrds\\
&\sim \mathit{const.}\ t^{-1/2}\int_0^t\int_0^\infty\langle\rho,\varphi GT_u(T_{u+r}G_{t-r}\varphi)^2\rangle dudr && \text{(by l'Hôpital)}\\
&\sim \mathit{const.}\ t^{1/2}\int_0^t\int_0^\infty\langle\rho,\varphi GT_u(T_{u+r}G_{t-r}\varphi\cdot T_{u+r}T_{t-r}\varphi)\rangle dudr && \text{(by l'Hôpital)}\\
&\le \mathit{const.}\ t^{1/2-\delta}\int_0^t\int_0^\infty\langle\rho,\varphi GT_u(T_{u+r}G_{t-r}\varphi\cdot(t+u)^\delta T_{t+u}\varphi)\rangle dudr\\
&\le \mathit{const.}\ t^{3/2-\delta}||G^3\varphi||\langle\rho,\varphi\rangle && \text{(by (2.2.1))}\\
&\longrightarrow 0.
\end{aligned}
$$

$$
\begin{aligned}
K_{1,5}(t) &= t^{-3/2}\int_0^t\int_0^s\int_0^\infty\langle\rho,T_{s-r}G_r\varphi\cdot T_u(T_u\varphi\cdot T_uT_rG_{s-r}\varphi)\rangle dudrds\\
&\le t^{-3/2}\int_0^t\int_0^s\int_0^\infty\langle\rho,\varphi G^2T_u(T_u\varphi\cdot T_{u+r}G_{s-r}\varphi)\rangle dudrds\\
&\sim \mathit{const.}\ t^{-1/2}\int_0^t\int_0^\infty\langle\rho,\varphi G^2T_u(T_u\varphi\cdot T_{u+r}G_{t-r}\varphi)\rangle dudr && \text{(by l'Hôpital)}\\
&\sim \mathit{const.}\ t^{1/2}\int_0^t\int_0^\infty\langle\rho,\varphi G^2T_u(T_u\varphi\cdot T_{u+r}T_{t-r}\varphi)\rangle dudr && \text{(by l'Hôpital)}\\
&\le \mathit{const.}\ t^{1/2-\delta}\int_0^t\int_0^\infty\langle\rho,\varphi G^2T_u(T_u\varphi\cdot(t+u)^\delta T_{t+u}\varphi)\rangle dudr\\
&\le \mathit{const.}\ t^{3/2-\delta}||G^3\varphi||\langle\rho,\varphi\rangle && \text{(by (2.2.1))}\\
&\longrightarrow 0.
\end{aligned}
$$

$K_{2,1}(t) \to 0$, similarly to $K_{1,1}(t) \to 0$.

$K_{2,2}(t) \to 0$, similarly to $K_{1,3}(t) \to 0$.

$$\begin{aligned}
K_{3,1}(t) &\leq ||G\varphi||t^{-3/2} \int_0^t \int_0^s \langle \rho, \varphi r T_r G\varphi \rangle drds \\
&\leq const. ||G\varphi||\ ||G^3\varphi|| \langle \rho, \varphi \rangle t^{-1/2} \to 0.
\end{aligned}$$

$$\begin{aligned}
K_{3,2}(t) &= t^{-2/3} \int_0^t \int_0^s \int_0^r \langle \rho, \varphi G T_u (T_{r-u} G_{s-r} \varphi)^2 \rangle dudrds \\
&\sim const.\ t^{-1/2} \int_0^t \int_0^r \langle \rho, \varphi G T_u (T_{r-u} G_{t-r} \varphi)^2 \rangle dudr \qquad \text{(by l'Hôpital)} \\
&\sim const.\ t^{1/2} \int_0^t \int_0^r \langle \rho, \varphi G T_u (T_{r-u} G_{t-r} \varphi \cdot T_{r-u} \varphi) \rangle dudr \qquad \text{(by l'Hôpital)} \\
&= const.\ t^{1/2} \int_0^t \int_0^r \langle \rho, \varphi G T_{r-u} (T_u G_{t-r} \varphi \cdot T_u T_{t-r} \varphi) \rangle dudr \\
&= const.\ t^{1/2} \int_0^r \int_0^{t-r} \langle \rho, \varphi G T_{t-r-u} (T_u G_r \varphi \cdot T_{u+r} \varphi) \rangle dudr \\
&\sim const.\ \ (M_1 H) + M_2(t)),
\end{aligned}$$

where (by l'Hôpital),

$$M_1(t) = t^{3/2} \int_0^t \langle \rho, \varphi G (T_{t-r} G_r \varphi \cdot T_t \varphi) \rangle dr$$

and, since $\dfrac{d}{dt} G T_t \varphi = -T_t$,

$$M_2(t) = -t^{3/2} \int_0^t \int_0^{t-r} \langle \rho, \varphi T_{t-r-u} (T_u G_r \varphi \cdot T_{u+r} \varphi) \rangle dudr.$$

Now,

$$\begin{aligned}
M_1(t) &= t^{3/2-\delta} \int_0^t \langle \rho, \varphi G (T_{t-r} G_r \varphi \cdot t^\delta T_t \varphi \rangle dr \\
&\leq const.\ t^{5/2-\delta} ||G^2\varphi|| \langle \rho, \varphi \rangle \qquad \text{(by (2.2.1))} \\
&\quad \longrightarrow 0,
\end{aligned}$$

$$\begin{aligned}
|M_2(t)| &\leq t^{3/2} \int_0^t \int_0^r \langle \rho, \varphi T_{r-u} (T_u G_{t-r} \varphi \cdot T_{t-(r-u)} \varphi \rangle dudr \\
&= t^{3/2} \int_0^t \int_0^r \langle \rho, \varphi T_u (T_{r-u} G_{t-r} \varphi \cdot T_{t-u} \varphi) \rangle dudr \\
&= t^{3/2} \int_0^t \left\langle \rho, \varphi T_u \left(\int_u^t T_{r-u} G_{t-r} \varphi dr \cdot T_{t-u} \varphi \right) \right\rangle du \\
&\leq ||G^2\varphi|| t^{5/2} \langle \rho, \varphi T_t \varphi \rangle \\
&= ||G^2\varphi|| t^{5/2-\delta} \langle \rho, \varphi t^\delta T_t \varphi \rangle \\
&\leq const.\ ||G^2\varphi|| \langle \rho, \varphi \rangle t^{5/2-\delta} \qquad \text{(by (2.2.1))} \\
&\quad \longrightarrow 0.
\end{aligned}$$

Finally, the weak convergence of Y_t as $t \to \infty$ follows as in the 1–level case. $\square$

5.3. Examples

Proof of Lemma 3.1.1:

α–stable process:

All the proofs for the α–stable case can be done by using the self–similarly of the transition probability p_t. We will prove only (c) and (d) to exemplify.

(c)
$$G_t\varphi(x) = \int_0^t \int_{\mathbb{R}^d} p_s(x-y)\varphi(y)dyds = \int_0^t s^{-d/\alpha} \int_{\mathbb{R}^d} p_1(s^{-1/\alpha}(x-y))\varphi(y)dyds$$
$$= t^{-d/\alpha+1} \int_0^1 r^{-d/\alpha} \int_{\mathbb{R}^d} p_1(t^{-1/\alpha}r^{-1/\alpha}(x-y))\varphi(y)dydr,$$

hence
$$\lim_{t\to\infty} t^{1-d/\alpha} G_t\varphi(x) = \frac{1}{1-d/\alpha} p_1(0) \int_{\mathbb{R}^d} \varphi(y)dy,$$

and (by Fourier transform)
$$p_1(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|z|^\alpha} dz.$$

(d)
$$G'_t\varphi(x) = \int_{\mathbb{R}^d} p_t(x-y)\varphi(y)dy = t^{-1} \int_{\mathbb{R}^d} p_1(t^{-1/\alpha}(x-y))\varphi(y)dy,$$

hence
$$\lim_{t\to\infty} tG'_t\varphi(x) = p_1(0) \int_{\mathbb{R}^d} \varphi(y)dy,$$

with
$$p_1(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|z|^d} dz,$$

and the result follows by l'Hôpital's rule.

Hierarchical random walk:

Recall that $a^{\gamma+1} = \dfrac{1}{N}$, by (4.2.3), and the definitions of θ and the functions h in Subsection 4.2 (see (4.2.4)–(4.2.9)).

(a) By (3.1.3), up to a summand which converges to 0 exponentially fast as $t \to \infty, p_t(0,x)$ equals

$$(N-1)\sum_{j=1}^{\infty} \frac{1}{N^j} e^{-\theta a^j t} \quad = \quad (N-1)(\theta t)^{-(\gamma+1)} \sum_{j=1}^{\infty} e^{-\theta a^j t}(\theta a^j t)^{\gamma+1}$$

$$= \quad q_\gamma t^{-(\gamma+1)}\Big(h_t^{(1,\gamma+1)} - \sum_{j\le 0} e^{-\theta a^j t}(\theta a^j t)^{\gamma+1}\Big).$$

To conclude the proof of (a) if suffices to observe that $\sum_{j\le 0} e^{-\theta a^j t}(\theta a^j t)^{\gamma+1} \to 0$ as $t\to\infty$. Indeed for all t and all negative integers j we have

$$(\theta a^j t)^{\gamma+1} e^{-\theta a^j t} \le const. \frac{1}{\theta a^j t},$$

hence $\sum\limits_{j\le 0} e^{-\theta a^j t}(\theta a^j t)^{\gamma+1}$ is majorized by $\dfrac{1}{t}$ times a convergent geometric series.

(b) Because of (a) it suffices to compute

$$\begin{aligned}\int_t^\infty s^{-(\gamma+1)} h_s^{(1,\gamma+1)} ds &= \int_t^\infty \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma+1} e^{-\theta a^j s} ds \\ &= \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma} e^{-\theta a^j t} = t^{-\gamma} h_t^{(1,\gamma)}.\end{aligned}$$

(c) Since $G_t(0,x) = \int_0^t p_s(0,x)ds$, we infer from (a) that

$$\begin{aligned}G_t(0,x) &\sim \int_0^t s^{-(\gamma+1)} h_s^{(1,\gamma+1)} ds \\ &= q_\gamma \int_0^t \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma+1} e^{-\theta a^j s} ds \\ &= q_\gamma \sum_{j-\infty}^{\infty} (\theta a^j)^{\gamma}(1-e^{-\theta a^j t}) \\ &= q_\gamma t^{-\gamma} h_t^{(2,\gamma)}.\end{aligned}$$

(d) Since $a = \dfrac{1}{N}$ for $\gamma = 0$, up to a summand which is uniformly bounded in t, $G_t(0,x)$ equals

$$\frac{N-1}{\theta}\sum_{j=1}^{\infty}(1-e^{-\theta a^j t}).$$

Since $a^{j+1} \le a^y \le a^i$ for $y \in [j, j+1]$, then

$$\int_1^\infty (1-e^{-a^y t})dy \le \sum_{j=1}^{\infty}(1-e^{-a^j t}) \le \int_1^\infty (1-e^{-a^y a^{-1} t})dy. \tag{5.3.1}$$

Now,

$$\int_1^\infty (1-e^{-a^y t})dy = \frac{1}{-\log a}\int_0^a \frac{1-e^{-zt}}{z}dz = \frac{1}{-\log a}\int_0^{at} \frac{1-e^{-r}}{r}dr,$$

and by l'Hôpital's rule,

$$\int_0^{at} \frac{1-e^{-r}}{r}dr \sim \log t.$$

Hence it follows from (5.3.1) that

$$\sum_{j=1}^{\infty}(1-e^{-a^j t}) \sim \frac{\log t}{-\log a},$$

which implies the result.

(e) Since $G_t^2(0,x) = \int_0^t \int_0^t p_{u+v}(0,x)dudv$, we infer from (a) that

$$\begin{aligned} G_t^2(0,x) &\sim q_\gamma \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma+1} \int_0^t \int_0^t e^{-\theta a^j (u+v)} dudv \\ &= q_\gamma \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma-1} (1-e^{-\theta a^j t})^2 \\ &= q_\gamma t^{1-\gamma} h_t^{(3,\gamma-1)}. \end{aligned}$$

(f) Since $a = \dfrac{1}{N^{1/2}}$ for $\gamma = 1$, up to a summand which is uniformly bounded in t, $G_t^2(0,x)$ equals

$$\frac{N-1}{\theta^2} \sum_{j=1}^{\infty} (1-e^{-\theta a^j t})^2.$$

The same argument used for (d) shows that

$$\sum_{j=1}^{\infty}(1-e^{-a^j t})^2 \sim \frac{-\log t}{\log a},$$

and the result follows.

(g) Since $G_t^2 G(0,x) = \int_0^\infty \int_0^t \int_0^t p_{s+u+v}(0,x)dudvds$, we infer from (a) that

$$\begin{aligned} G_t^2 G(0,x) &\sim q_\gamma \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma+1} \int_0^\infty \int_0^t \int_0^t e^{-\theta a^j (s+u+v)} dudvds \\ &= q_\gamma \sum_{j=-\infty}^{\infty} (\theta a^j)^{\gamma-2} (1-e^{-\theta a^j t})^2 \\ &= q_\gamma t^{2-\gamma} h_t^{(3,\gamma-2)}. \end{aligned}$$

(h) Since $a = \dfrac{1}{N^{1/3}}$ for $\gamma = 2$, up to a summand which is uniformly bounded in t, $G_t^2 G(0,x)$ equals

$$\frac{N-1}{\theta^3} \sum_{j=1}^{\infty} (1-e^{-\theta a^j t})^2,$$

and the proof is the same as for (f). □

Proof of Lemma 3.1.2:

By the observation before Lemma 2.4.1, $||G^j \varphi|| < \infty$ for $\mathcal{C}_c^+(S)$ if and only if $j < \gamma + 1$, i.e. $\alpha j < d$ for the α–stable case, and $c > N^{(j-1)/j}$ for the c–hierarchical case.

The expression for G^j can be obtained by formula (2.1.1). For the c–hierarchical case the form of the semigroup T_t is explicit from the transition probability $p_t(0,x)$ given in (3.1.3). For the α–stable case the transition probability $p_t(0,x)$ is not known explicitly for general α, but its Fourier transform,

$$\int_{\mathbb{R}^d} e^{-ix\cdot z} p_t(0,x)dx = e^{-t|z|^\alpha},$$

can be used for the proof. $\square$

Proof of Lemma 3.1.3:

We sketch the main idea of the proof. By Lemma 3.1.1(a), $T_t \sim t^{-(\gamma+1)}$. By Lemma 2.4.1, $\gamma > k+1$. Hence $T_t = o(t^{-(k+2)})$. $\square$

Proof of Proposition 3.1.1:

Let h_t denote any of the functions defined in (4.2.5) – (4.2.7). We write h_t as $h_t = h_t^{(-)} + h_t^{(+)}$, where $h_t^{(-)}$ and $h_t^{(+)}$ stand for the sums $\sum\limits_j$ with $j < 0$ and $j \geq 0$, respectively. Since $\lim_{t\to\infty} h_t^{(-)} = 0$ (as in the proof of Lemma 3.1.1 for the hierarchical case), and h_t is periodic in a logarithmic scale, in order to prove that

$$L_1 \leq \inf_t h_t \leq \sup_t h_t \leq L_2,$$

for some positive constants L_1 and L_2, it suffices to show that

$$L_1 \leq \liminf_{t\to\infty} h_t^{(+)} \leq \limsup_{t\to\infty} h_t^{(+)} \leq L_2. \tag{5.3.2}$$

We will prove (5.3.2) for $h_t^{(1,\zeta)}$. Using the formula

$$q^{-j} = \frac{q\log q}{q-1}\int_j^{j+1} q^{-y}dy, q>0, \quad q\neq 1,$$

and $a^{j+1} \leq a^y \leq a^j$ for $j \leq y \leq j+1$ (since $0<a<1$), we obtain

$$\frac{a^{-\zeta}\log a^{-\zeta}}{a^{-\zeta}-1}\int_0^\infty a^{y\zeta}e^{-a^{-y}a^{-1}t}dy \leq \sum_{j=0}^\infty a^{j\zeta}e^{-a^j t} \leq \frac{a^{-\zeta}\log a^{-\zeta}}{a^{-\zeta}-1}\int_0^\infty a^{y\zeta}e^{-a^y t}dy.$$

We have

$$\int_0^\infty a^{y\zeta}e^{-a^y t}dy = \frac{1}{-\log a}\int_0^1 z^{\zeta-1}e^{-zt}dz = \frac{t^{-\zeta}}{-\log a}\int_0^t r^{\zeta-1}e^{-r}dr,$$

and since $\int_0^t r^{\zeta-1}e^{-r}dr \to \Gamma(\zeta)$ as $t\to\infty$, putting these results together we obtain

$$\frac{\Gamma(\zeta+1)}{a^{-\zeta}-1} \leq \liminf_{t\to\infty} t^\zeta \sum_{j=0}^\infty a^{j\zeta}e^{-a^j t} \leq \limsup_{t\to\infty} t^\zeta \sum_{j=0}^\infty a^{j\zeta}e^{-a^j t} \leq \frac{a^{-\zeta}\Gamma(\zeta+1)}{a^{-\zeta}-1},$$

which finishes the proof.

This method can be used for the other functions h_t as well, with slightly more elaborate calculations. For $\widetilde{h}_t^{(2,\zeta)}$ and $\widetilde{h}_t^{(3,\zeta)}$ we need to use the fact that the functions $x \mapsto e^{-x}-1+x$ and $x \mapsto 2e^{-x}-\frac{1}{2}e^{-2x}+x-\frac{3}{2}$, respectively, are increasing. However, we can obtain bounds for $\widetilde{h}_t^{(2,\zeta)}$ and $\widetilde{h}_t^{(3,\zeta)}$ from the bounds for $h_t^{(2,\zeta)}$ and $h_t^{(3,\zeta)}$, simply by dividing them by $1-\zeta$. This is clear from the form of $\widetilde{h}_t$ in Lemma 2.4.3. □

Proof of Theorem 3.2.1:

The proof is a direct application of Theorems 2.2.1, 2.2.1, 2.2.3, Lemmas 3.1.1, 3.1.2, 3.1.3, and Corollary 3.1.1. □

5.4. Conditions for the results on infinite variance branching results

Proof of Porposition 3.2.1: We have to show that

$$d > \alpha\left(1+\frac{1}{\beta}\right) \tag{5.4.1}$$

is necessary and sufficient for condition (2.5.1) of Theorem 2.5.1.

That (5.4.1) implies (2.5.1) is proved by Iscoe[27]. For the converse, note that

$$G1_B(x) \geq k|x|^{-(d-\alpha)} \text{ for } |x| \geq 2,$$

where B denotes the unit ball centered at the origin and k is some positive constant. Hence, if $d \leq \alpha(1+1/\beta), (G1_B)^{1+\beta}$ is not λ–integrable. □

For the proof of Proposition 3.2.2 we need some preliminary results.

Let R_∞^1 and R_∞ be the canonical measure of the equilibrium of the particle system (started off in the Poisson system Π_λ with intensity λ) and that of the superprocess (started off in λ), respectively (Appendix, Subsection A.1).

Lemma 5.4.1. Assume that $\beta_2 < \beta_1$ and let $\varphi : \mathbb{R}^d \to [0,\infty]$. Then
(a)

$$\int_{\mathcal{M}_\tau(S)} \langle \nu, \varphi\rangle^{1+\beta_2} R_\infty(d\nu) \leq \int_{\mathcal{M}_\tau(S)} \langle \nu, \varphi\rangle^{1+\beta_2} R_\infty^1(d\nu).$$

b) If φ is λ-integrable, then

$$\int_{\mathcal{M}_\tau(S)} \langle \nu, \varphi\rangle^{1+\beta_2} R_\infty^1(d\nu) \leq C < \infty,$$

where the constant C depends only on $d, \alpha, \beta_1, \beta_2$ and $\langle \lambda, \varphi\rangle$.

Proof: (a) We have, by Jensen's inequality and (A.1.10),

$$\begin{aligned}\int \langle \nu, \varphi \rangle^{1+\beta_2} R_\infty (d\nu) &= \int \left(\int \langle \mu, \varphi \rangle \Pi_\nu (d\mu) \right)^{1+\beta_2} R_\infty (d\nu) \\ &\leq \int \int \langle \mu, \varphi \rangle^{1+\beta_2} \Pi_\nu (d\mu) \, R_\infty (d\nu) \\ &= \int \langle \mu, \varphi \rangle^{1+\beta_2} R^1_\infty (d\nu) \, .\end{aligned}$$

(b) By the Palm formula (A.2.1) it suffices to show for some $\delta > \beta_2$,

$$\int \langle \nu, \varphi \rangle^\delta (R^1_\infty)_x (d\nu) < C < \infty,$$

where the constant C does not depend on x. We can choose $\delta \in (\beta_2, \beta_1)$ such that $d\beta_2/\alpha > 1$.

We will use the tree representation of $(R^1_\infty)^{red}_x$ given in (A.3.2), and we denote $Z_{t,i} = \langle X_t^{W^x_t, i}, \varphi \rangle$. Since $\left(\sum_{j=1}^n a_j \right)^\delta \leq \sum_{j=1}^n a_j^\delta$, for all nonnegative sequences (a_j) and $0 < \delta < 1$, we have

$$\begin{aligned}\int \langle \nu, \varphi \rangle^\delta \left(R^1_\infty \right)^{red}_x (d\nu) &= E\biggl(\biggl(\int_0^\infty \biggl(\sum_{i=1}^{N_t} Z_{t,i} \biggr) \pi(dt) \biggr)^\delta \biggr) \\ &= E\biggl(E\biggl[\biggl(\int_0^\infty \biggl(\sum_{i=1}^{N_t} Z_{t,i} \; biggr) \pi(dt) \biggr)^\delta \bigg| \pi, W^x \biggr] \biggr) \\ &\leq E\biggl(E\biggl[\biggl(\int_0^\infty \biggl(\sum_{i=1}^{N_t} Z_{t,i} \; biggr)^\delta \pi(dt) \biggr) \bigg| \pi, W^x \biggr] \biggr) \\ &= E\biggl(\int_0^\infty E\biggl[\biggl(\sum_{i=1}^{N_t} Z_{t,i} \biggr)^\delta \bigg| W^x \biggr] \pi(dt) \biggr).\end{aligned}$$

Now, by the self–similarity of the α–stable process,

$$\begin{aligned}E\left[Z_{t,i} | W^x \right] &= \int \varphi \left(y \right) p_t \left(W^x_t, y \right) \lambda \left(dy \right) \\ &= \int \varphi \left(y - W^x_t \right) p_t \left(0, y \right) \lambda \left(dy \right) \\ &= \frac{1}{t^{d/\alpha}} \int \varphi \left(y - W^x_t \right) p_1 \left(0, y t^{-1/\alpha} \right) dy \\ &\leq K \frac{1}{t^{d/\alpha}} \langle \lambda, \varphi \rangle,\end{aligned}$$

for some real constant K not depending on W^x and φ. Hence, by Hölder's inequality,

$$\begin{aligned}E\biggl[\biggl(\sum_{i=1}^{N_t} Z_{t,i} \biggr)^\delta \bigg| W^x \biggr] &= \sum_{n=0}^\infty E\biggl[\biggl(\sum_{i=1}^{n} Z_{t,i} \biggr)^\delta \bigg| W^x \biggr] P[N_t = n] \\ &= \sum_{n=0}^\infty n^\delta \; E\biggl[\biggl(\frac{1}{n} \sum_{i=1}^{n} Z_{t,i} \biggr)^\delta \bigg| W^x \; biggr] P[N_t = n]\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{\infty} n^{\delta} \left(E\left[\frac{1}{n} \sum_{i=1}^{n} Z_{t,i} \middle| W^x \right] \right)^{\delta} P\left[N_t = n\right] \\
&\leq \left(K\langle \lambda, \varphi\rangle\right)^{\delta} t^{-d\delta/\alpha} \sum_{n=0}^{\infty} n^{\delta} P\left[N_t = n\right].
\end{aligned}
$$

For $(1+\beta_1)$–branching and $\delta < \beta_1$,

$$
\sum_{n=0}^{\infty} n^{\delta} P\left[N_t = n\right] < \infty.
$$

On the other hand, since $d\delta/\alpha > 1$, we have

$$
E\left(\int_1^{\infty} t^{-d\delta/\alpha} \pi\left(dt\right) \right) < \infty.
$$

Putting these results together finishes the proof. □

Corollary 5.4.1. In the setting of Lemma 5.4.1(b),

$$
\int \langle \nu, \varphi\rangle^{\beta_2} (R_\infty)_x(d\nu) < \infty, \qquad x \in \mathbb{R}^d.
$$

Proof: Let ψ be strictly positive and λ-integrable. Then, by the Palm formula (A.2.1),

$$
\begin{aligned}
\int\int \langle \nu, \varphi\rangle^{\beta_2} (R_\infty)_x(d\nu)\psi(x)\lambda(dx) &= \int \langle \nu, \varphi\rangle^{\beta_2} \langle \nu, \psi\rangle R_\infty(d\nu) \\
&\leq \int \langle \nu, \varphi + \psi\rangle^{1+\beta_2} R_\infty(d\nu) < \infty.
\end{aligned}
$$

This shows that the assertion of the corollary holds for λ–almost all x. The shift–invariance of the system implies that it is in fact true for all $x \in \mathbb{R}^d$. □

Lemma 5.4.2.

$$
(R^1_\infty)_x = \delta_{\delta_x} * \int_{\mathcal{M}_\tau(S)} \Pi_\nu(\cdot)(R_\infty)_x(d\nu) \ \text{ for } \lambda - \text{almost all } x.
$$

Proof: Since R^1_∞ has intensity measure λ, by (A.1.11), the assertion is obtained from the following chain of equalities, where we use the Palm formula (A.2.1) for Π_ν and for R_∞, the fact that $(\Pi_\nu)_x = \delta_{\delta_x} * \Pi_\nu$, and (A.1.10),

$$
\begin{aligned}
&\int\int\int f(x)\, F(\mu)(\delta_{\delta_x} * \Pi_\nu)(d\mu)(R_\infty)_x(d\nu)\lambda(dx) \\
&= \int\int\int f(x)F(\mu)(\delta_{\delta_x} * \Pi_\nu)(d\mu)\nu(dx)R_\infty(d\nu) \\
&= \int\int F(\mu) \int f(x)\mu(dx)\Pi_\nu(d\mu)R_\infty(d\nu) \\
&= \int F(\mu) \int f(x)\mu(dx)R^1_\infty(d\mu).
\end{aligned}
$$

□

Proof of Proposition 3.2.2: We have to prove that

$$\beta_2 < \beta_1 < 1 \tag{5.4.2}$$

and

$$d > \alpha\left(1 + \frac{1}{\beta_2}\left(1 + \frac{1}{\beta_1}\right)\right). \tag{5.4.3}$$

are necessary and sufficient for condition (2.5.2) of Theorem 2.5.2.

1. Assume that (5.4.2) and (5.4.3) hold. Because of Lemma 3.1.2 and (4.1.2), we have to show that

$$\int \left(\iint \frac{\varphi(y)}{|x-y|^{d-\alpha}} \; dy \; \mu(dx)\right)^{1+\beta_2} R^1_\infty(d\mu) < \infty, \quad \varphi \in \mathcal{C}_c^+(\mathbb{R}^d).$$

First we observe that (Iscoe[(27)], Lemma 5.3)

$$\int \frac{\varphi(y)}{|x-y|^{d-\alpha}} \; dy \leq const.(1 \vee |x|)^{-d+\alpha}. \tag{5.4.4}$$

Hence from Lemma 5.4.1(b) (denoting by B_r the ball with radius r centered at 0) we obtain

$$\int \left(\int_{B_1} \int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-\alpha}} \; dy \; \mu(dx)\right)^{1+\beta_2} R^1_\infty(d\mu) < \infty,$$

and it suffices to show that

$$A := \int \left(\int_{\mathbb{R}^d} 1_{B_1^c}(x) \; |x|^{-d+\alpha} \; \mu(dx)\right)^{1+\beta_2} R^1_\infty(d\mu) < \infty.$$

Using the Palm formula (A.2.1) and Lemma 5.4.2, we have

$$\begin{aligned}
A &= \int 1_{B_1^c}(x)\,|x|^{-d+\alpha} \int \left(\int 1_{B_1^c}(z)\,|z|^{-d+\alpha}\,\mu(dz)\right)^{\beta_2} (R^1_\infty)_x(d\mu)\lambda(dx) \\
&= \int 1_{B_1^c}(x)\,|x|^{-d+\alpha} \int\int \left(1_{B_1^c}(x)\,|x|^{-d+\alpha} + \int 1_{B_1^c}(z)\,|z|^{-d+\alpha}\,\mu(dz)\right)^{\beta_2} \\
&\qquad\qquad \cdot \Pi_\nu(d\mu)(R_\infty)_x(d\nu)\lambda(dx) \\
&\leq \int 1_{B_1^c}(x)\,|x|^{-d+\alpha}(|x|^{-d+\alpha})^{\beta_2}\lambda(dx) \\
&\quad + \int 1_{B_1^c}(x)\,|x|^{-d+\alpha} \int\int \left(\int 1_{B_1^c}(z)\,|z|^{-d+\alpha}\mu(dz)\right)^{\beta_2} \Pi_\nu(d\mu)(R_\infty)_x(d\nu)\lambda(dx).
\end{aligned}$$

Since $\alpha + \beta_2(-d+\alpha) < 0$ by (5.4.3), the first term on the r.h.s. is finite.

Using Hölder's inequality, the second term can be bounded by

$$B := \int_{B_1^c} |x|^{-d+\alpha} \int \left(\int |z|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_x(d\nu)\lambda(dx),$$

and by shift–invariance,

$$B = \int_{B_1^c} |x|^{-d+\alpha} \int \left(\int |z-x|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu)\lambda(dx).$$

The scaling property of $(R_\infty)_0$ (Dawson and Perkins[11], Theorem 6.7) yields

$$\begin{aligned} B &= \int_{B_1^c} |x|^{-d+\alpha}|x|^{\alpha/\beta_1} \int \left(\int |\, z|x| \, - \, x\,|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu)\lambda(dx) \\ &= \int_{B_1^c} |x|^{-d+\alpha+\alpha/\beta_1+\beta_2(-d+\alpha)} \int \left(\int \left| z - \frac{x}{|x|} \right|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu)\lambda(dx). \end{aligned}$$

By isotropy of $(R_\infty)_0$, the integral w.r. to $(R_\infty)_0$ does not depend on $x/|x|$. For an arbitrary fixed $x_0 \in \mathbb{R}^d$ we put $e = x_0/|x_0|$ and we obtain

$$B = \int_{B_1^c} |x|^{-d+\alpha+\alpha/\beta_1+\beta_2(-d+\alpha)}\lambda(dx) \int \left(\int |z-e|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu).$$

Since $\alpha + \alpha/\beta_1 + \beta_2(-d+\alpha) < 0$ by (5.4.3), the integral w.r. to λ is finite. Hence we will be done if we can show that the integral w.r. to $(R_\infty)_0$ is finite as well.

We will show that

$$C_1 := \int \left(\int_{B_2} |z-e|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\mu)$$

and

$$C_2 := \int \left(\int_{B_2^c} |z-e|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\mu)$$

are finite.

Let

$$g(z) = 1_{B_2}(z)|z-e|^{-d+\alpha}.$$

By Corollary 5.4.1 we have

$$C_1 = \int \left(\int g(z)\nu(dz) \right)^{\beta_2} (R_\infty)_0(d\mu) < \infty,$$

since $\langle \lambda, g \rangle < \infty$. On the other hand, for $z \in B_2^c$, $|z-e| \geq \frac{1}{2}|z|$. Therefore

$$C_2 \leq const. \int \left(\int_{B_2^c} |z|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu).$$

To show the finiteness of the latter integral we will decompose the random variable $\int_{B_2^c} |z|^{-d+\alpha}\nu(dz)$ into a sum of terms whose L^{β_2}–norms add up to something finite. Put $D_k := B_{2^{k+1}} \backslash B_{2^k}$, $k = 0, 1, \ldots$

Then, again by the scaling property of $(R_\infty)_0$ we have

$$\begin{aligned}
&\int \left(\int_{D_k} |z|^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu) \\
&= (2^k)^{\alpha/\beta_1} \int \left(\int_{D_0} (2^k|z|)^{-d+\alpha} \nu(dz) \right)^{\beta_2} (R_\infty)_0(d\nu) \\
&\le 2^{k(\alpha/\beta_1+\beta_2(-d+\alpha))} \int (\nu(D_0))^{\beta_2} (R_\infty)_0(d\nu).
\end{aligned}$$

The L^{β_2}–norm of the k–th summand is thus bounded by $const.2^{k(\alpha/\beta_1\beta_2-d+\alpha)}$, which is summable since $\alpha/\beta_1\beta_2 - d + \alpha < 0$ due to (5.4.3).

2. To prove the converse we assume that (2.5.2) holds.

(a) Assume that $\beta_2 \ge \beta_1$, $\beta_1 < 1$. We will show that, for $\varphi \in C_c(S)$, $\varphi \ge 0$, $\varphi \ne 0$,

$$\int \langle \mu, G\varphi \rangle^{1+\beta_2} R^1_\infty(d\mu) = \infty.$$

First note that by the Palm formula (A.2.1),

$$\int \langle \mu, G\varphi \rangle^{1+\beta_2} R^1_\infty(d\mu) = \int G\varphi(x) \int \langle \mu, G\varphi \rangle^{\beta_2} (R^1_\infty)_x(d\mu) \lambda(dx). \tag{5.4.5}$$

Choose $\psi : \mathbb{R}^d \to \mathbb{R}_+$ such that $G\varphi \ge \psi(\cdot - x)$ provided that $|x| \le 1$. Then the r.h.s. of (5.4.5) is bounded above by

$$\int 1_{\{|x|\le 1\}} G\varphi(x) \int \langle \mu, \psi \rangle^{\beta_2} (R^1_\infty)_0(d\mu) \lambda(dx).$$

By the tree representation (A.3.2) of $(R^1_\infty)_0$ we have

$$\int \langle \mu, \psi \rangle^{\beta_2} (R^1_\infty)_0(d\mu) \ge E\left(\left(\sum_{i=1}^{N} \langle X_\sigma^{W^0_\sigma, i}, \psi \rangle \right)^{\beta_2} \right), \tag{5.4.6}$$

where σ is exponentially distributed with parameter V_1, and N is distributed like any of the N_t. Since $EN^{\beta_2} = \infty$, and since, conditioned on σ and W^0, the random variables $\langle X_\sigma^{W^0_\sigma, i}, \psi \rangle$ are i.i.d. with positive expectation, it follows from the law of large numbers that the r.h.s. of (5.4.6) is infinite.

(b) Now assume that $1 \ge \beta_1 > \beta_2$, and suppose that $\varphi(x) \ge 1$ for $|x| \le 1$. Then, for some $k > 0$,

$$G\varphi(x) \ge k|x|^{-(d-\alpha)} \qquad \text{if } |x| \ge 2.$$

Assume that $\int \langle \mu, G\varphi \rangle^{1+\beta_2} R^1_\infty(d\mu) < \infty$. Then, by Lemma 5.4.1(a),

$$\int \langle \nu, G\varphi \rangle^{1+\beta_2} R^1_\infty(d\nu) < \infty.$$

Therefore, by the Palm formula (A.2.1) we have

$$\begin{aligned}\infty &> \int \langle \nu, G\varphi\rangle^{1+\beta_2} R^1_\infty(d\nu) \\ &\geq k^{1+\beta_2} \int 1_{\{|x|\geq 1\}}|x|^{-d+\alpha} \int \left(\int 1_{\{|z|\geq 1\}}|z|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_x(d\nu)\lambda(dx).\end{aligned}$$

By shift–invariance this equals

$$k^{1+\beta_2} \int 1_{\{|x|\geq 1\}}|x|^{-d+\alpha} \int \left(\int 1_{\{|z-x|\geq 1\}}|z-x|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_0(d\nu)\lambda(dx).$$

The scaling property of $(R_\infty)_0$ permits to rewrite the inner integral (with $e(x) = x/|x|$) as

$$|x|^{\alpha/\beta_1} \int \left(\int 1_{\{|\, z\,|x| - x\,| \,\geq 1\}}|\, z\,|x| \;-\; x\,|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_0(d\nu)$$

$$= |x|^{\alpha/\beta_1+\beta_2(-d+\alpha)} \int \left(\int 1_{\{|x||z-e(x)|\geq 1\}}|z-e(x)|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_0(d\nu).$$

For $|x| \geq 1$, the latter integral is bounded below by

$$\int \left(\int 1_{\{|z|\geq 1\}}|2z|^{-d+\alpha}\nu(dz)\right)^{\beta_2} (R_\infty)_0(d\nu) > 0.$$

Now,

$$\int |x|^{-d+\alpha+\alpha/\beta_1+\beta_2(-d+\alpha)} 1_{\{|x|\geq 1\}}\lambda(dx) < \infty$$

implies that $\alpha + \alpha/\beta_1 + \beta_2(-d+\alpha) < 0$, or equivalently, (5.4.3) holds. □

APPENDIX

A.1. Background on 1- and 2-level branching systems

We consider particle systems in a locally compact Abelian group S with Haar measure ρ. Recall that T_t denotes the semigroup of the particle motion and G the corresponding Green operator.

Let $\mathcal{C}(S)$ denote the space of bounded continuous functions on S, $\mathcal{C}_0(S)$ the subspace of functions vanishing at infinity, and $\mathcal{C}_c(S)$ that of functions with compact support. For a strictly positive function $\tau \in \mathcal{C}_0(S)$, let

$$\mathcal{C}_\tau(S) = \{\varphi \in \mathcal{C}(S) : \varphi\tau^{-1} \in \mathcal{C}_0(S)\}$$

with the norm $||\varphi||_\tau = ||\varphi\tau^{-1}||$. We assume that τ is such that $t \mapsto T_t\varphi$ is a continuous curve in $(\mathcal{C}_\tau(S), ||\cdot||_\tau)$ for each $\varphi \in \mathcal{C}_\tau(S)$. For example, in the case of the α-stable motion in $\mathbb{R}^d$ we may take

$\tau(x) = (1+|x|^2)^{-q}$ with $d/2 < q < (d+\alpha)/2$ (Dawson and Gorostiza[6]). The subspaces of non–negative functions are indicated with the superscript '+', e.g. $\mathcal{C}_\tau^+(S)$. Let $\mathcal{M}_\tau(S)$ denote the space of non–negative Radon measures μ on S such that $\langle \mu, \tau\rangle < \infty$, endowed with the smallest topology which makes the maps $\mu \mapsto \langle \mu, \varphi\rangle$ continuous for all $\varphi \in \mathcal{C}_c^+(S) \cup \{\tau\}$. We assume that $\rho \in \mathcal{M}_\tau(S)$. The subspace of $\mathcal{M}_\tau(S)$ of integer–valued measures is designated by $\mathcal{N}_\tau(S)$.

The Laplace functional of the occupation time of the 1–level branching particle system X_t with $(1+\beta)$–branching at rate V and started off from a Poisson system with intensity ρ is given by

$$E\exp\left\{-\left\langle \int_0^t X_s ds, \varphi\right\rangle\right\} = \exp\{-\langle \rho, u_\varphi(t)\rangle\},\ \varphi \in \mathcal{C}_\tau^+(S), \tag{A.1.1}$$

where $u_\varphi(x,t)$ with values in $[0,1]$ is the unique solution of the non–linear evolution equation

$$u_\varphi(t) = -\frac{V}{1+\beta}\int_0^t T_{t-s}(u_\varphi(s)^{1+\beta})ds + \int_0^t T_{t-s}(\varphi(1-u_\varphi(s)))ds. \tag{A.1.2}$$

This is shown by the same argument of Theorem 5 in Gorostiza and López–Mimbela[18] (formulas (4.8) and (4.9)). It follows that

$$u_\varphi(t) \le \int_0^t T_{t-s}(\varphi(1-u_\varphi(s)))ds \le G_t\varphi. \tag{A.1.3}$$

Let U_t denote the semigroup of the 1–level branching particle system. We have

$$U_t(\langle\cdot,\varphi\rangle)(\mu) = \langle \mu, T_t\varphi\rangle, \quad \varphi \in \mathcal{C}_\tau^+(S), \mu \in \mathcal{N}_\tau(S), \tag{A.1.4}$$

and, if $\beta = 1$,

$$\begin{aligned} U_t(\langle\cdot,\varphi\rangle\langle\cdot,\psi\rangle)(\mu) &= \langle \mu, T_t\varphi\rangle\langle\mu, T_t\psi\rangle + \langle \mu, T_t(\varphi\psi) - T_t\varphi\cdot T_t\psi\rangle \\ &+ V\int_0^t \langle \mu, T_s(T_{t-s}\varphi\cdot T_{t-s}\psi)\rangle ds, \quad \varphi,\psi \in \mathcal{C}_\tau^+(S),\ \mu \in \mathcal{N}_\tau \end{aligned} \tag{A.1.5}$$

The formulas (A.1.4) and (A.1.5) can be derived by martingale methods from the Markov property of the system (see e.g. Gorostiza and Rodrigues[20] for explicit calculations of this type). In particular,

$$U_t(\langle\cdot,\varphi\rangle)(\delta_x) = T_t\varphi(x), \tag{A.1.6}$$

$$U_t(\langle\cdot,\varphi\rangle\langle\cdot,\psi\rangle)(\delta_x) = T_t(\varphi\psi)(x) + V\int_0^t T_s(T_{t-s}\varphi\cdot T_{t-s}\psi)(x)ds. \tag{A.1.7}$$

We have from (A.1.4)

$$U_t(\langle\cdot,\varphi\rangle\langle\cdot,\psi\rangle)(\mu) \le \langle \mu, T_t\varphi\rangle\langle\mu, T_t\psi\rangle + \langle \mu, T_t(\varphi\psi)\rangle + V\int_0^t \langle \mu, T_s(T_{t-s}\varphi\cdot T_{t-s}\psi)\rangle ds,$$
$$\varphi,\psi \in \mathcal{C}_\tau^+(S). \tag{A.1.8}$$

If the 1–level branching system is persistent, it has a "Poisson type" equilibrium (in the sense of Liemant et al[31], section 2.3), which is an infinitely divisible random element of $\mathcal{M}_\tau(S)$. Its canonical

measure, which is a measure on $\mathcal{M}_\tau(S)$, is denoted by R^1_∞. A sufficient condition for persistence is

$$\int_0^\infty \langle \rho, (T_t\varphi)^{1+\beta}\rangle dt < \infty,\ \varphi \in \mathcal{C}_c^+(S)$$

(Gorostiza and Wakolbinger$^{(22)}$, Theorem 2.1).

For each $t > 0$, the random measure X_t is infinitely divisible and its canonical measure R_t has the form

$$R^1_t = \int_S P[X^x_t \in (\cdot)]\rho(dx), \tag{A.1.9}$$

where X^x_t corresponds to the branching system starting with a single ancestor in x at time 0 (Gorostiza and Wakolbinger$^{(21)}$, formula (3.1), Liemant et al$^{(31)}$).

The measure R^1_∞ is the "Poissonization" of the equilibrium canonical measure R_∞ of the superprocess counterpart of the particle system, i.e.,

$$R^1_\infty = \int_{\mathcal{M}_\tau(S)} \Pi_\nu(\cdot) R_\infty(d\nu), \tag{A.1.10}$$

where Π_ν is the distribution of a Poisson random measure on S with intensity measure ν. Indeed, in Gorostiza et al$^{(19)}$ it is shown that the distibution L^1_t of the branching particle system X_t is a Cox process, i.e.,

$$L^1_t = \int_{\mathcal{M}_\tau(S)} \Pi_\nu(\cdot) L_t(d\nu),$$

where L_t is the distribution of the superprocess counterpart of X_t. By continuity and the assumed persistence, this relation carries over to $t = \infty$:

$$L^1_\infty = \int_{\mathcal{M}_\tau(S)} \Pi_\nu(\cdot) L_\infty(d\nu).$$

Together with the Lévy-Khinchin formula (Kallenberg$^{(28)}$), this implies the following chain of equalities for each $\varphi \in C_c(S)$:

$$\begin{aligned}
\exp\left\{-\int R^1_\infty(d\mu)(1-e^{-\langle\mu,\varphi\rangle})\right\} &= \int e^{-\langle\mu,\varphi\rangle} L^1_\infty(d\mu) \\
&= \int\int e^{-\langle\mu,\varphi\rangle} \Pi_\nu(d\mu)\, L_\infty(d\nu) \\
&= \int e^{-\langle\nu,1-e^{-\varphi}\rangle} L_\infty(d\nu) \\
&= \exp\left\{-\int R_\infty(d\nu)\,(1-e^{-\langle\nu,1-e^{-\varphi}\rangle})\right\} \\
&= \exp\left\{-\int R_\infty(d\nu)\left(1-\int e^{-\langle\mu,\varphi\rangle}\Pi_\nu(d\mu)\right)\right\} \\
&= \exp\left\{-\int R_\infty(d\nu)\int \Pi_\nu(d\mu)\,(1-e^{-\langle\mu,\varphi\rangle})\right\},
\end{aligned}$$

which yields (A.1.10).

Let $\mathcal{M}^2_\tau(S)$ denote the space of Radon measures $\underline{\mu}$ on $\mathcal{M}_\tau(S)$ such that $\langle\!\langle \underline{\mu}, \langle\cdot,\tau\rangle\rangle\!\rangle < \infty$, where

$$\langle\!\langle \underline{\mu}, F\rangle\!\rangle = \int_{\mathcal{M}_\tau(S)} F(\nu)\underline{\mu}(d\nu)$$

(sometimes we use the notation on the r.h.s in order to avoid confusion). We have that $R^1_\infty \in \mathcal{M}^2_\tau(S)$, R^1_∞ is invariant (but not reversible) for the 1–level dynamics (Liemant et al[31], Chapter 2, Dawson and Perkins[11]), and it has intensity ρ in the sense that

$$\langle\!\langle R^1_\infty, \langle\cdot,\varphi\rangle\rangle\!\rangle = \langle\rho,\varphi\rangle, \quad \varphi\in\mathcal{C}^+_\tau(S). \tag{A.1.11}$$

If $\beta = 1$, then R^1_∞ has finite moments of all orders and the second and third moments are given by

$$\langle\!\langle R^1_\infty, \langle\cdot,\varphi\rangle\langle\cdot,\psi\rangle\rangle\!\rangle \quad = \quad \langle\rho,\varphi\psi\rangle + \frac{V}{2}\langle\rho,\varphi G\psi\rangle, \quad \varphi,\psi\in\mathcal{C}^+_c(S), \tag{A.1.12}$$

$$\begin{aligned}&\langle\!\langle R^1_\infty, \langle\cdot,\varphi\rangle\langle\cdot,\psi\rangle\langle\cdot,\zeta\rangle\rangle\!\rangle = \langle\rho,\varphi\psi\zeta\rangle + \frac{V}{2}\langle\rho,\varphi\psi G\zeta + \varphi\zeta G\psi + \psi\zeta G\varphi\rangle\\ &+\frac{V^2}{2}\left\langle\rho,\int_0^\infty[\varphi GT_t(T_t\psi\cdot T_t\zeta) + \psi GT_t(T_t\varphi\cdot T_t\zeta) + \zeta GT_t(T_t\varphi\cdot T_t\psi)]dt\right\rangle, \quad \varphi,\psi,\zeta\in\mathcal{C}^+_c(S).\end{aligned} \tag{A.1.13}$$

See Subsection A.4 for a proof.

Note also, from (A.1.9) and T_t–invariance of ρ, that for each $t > 0$,

$$\langle\!\langle R^1_t, \langle\cdot,\varphi\rangle\rangle\!\rangle = \langle\rho,\varphi\rangle,\ quad\varphi\in\mathcal{C}^+_\tau(S). \tag{A.1.14}$$

We pass now to the 2–level branching system. A "2–level particle" is an element μ of $\mathcal{N}_\tau(S)$ of the form $\mu = \sum_{i=1}^n \delta_{x_i}$. A "clan" is the progeny under the 1–level dynamics (i.e., individual particles undergoing $(1+\beta)$–branching at rate V_1) of a family of particles which constitute an initial 2–level particle. Clans undergo $(1+\beta_2)$–branching at rate V_2. Assuming persistence of the 1–level system, the 2–level system starts off from a Poisson system of "2–level particles" with intensity measure R^1_∞. The empirical measures of the 2–level system take values in $\mathcal{M}^2_\tau(S)$. Restricting to test functions on $\mathcal{M}_\tau(S)$ of the form $\mu \mapsto \langle\mu,\varphi\rangle$, $\varphi\in\mathcal{C}^+_c(S)$, amounts to considering the *aggregated* system, i.e., we consider the empirical measure of all the point masses disregarding which 2–level particles they belong to. Note that the moments of R^1_∞ in (A.1.11), (A.1.12), (A.1.13) correspond to the aggregated Poisson system. The empirical measures X_t of the aggregation of the 2–level system take values in $\mathcal{N}_\tau(S)$.

The same argument used to obtain the Laplace functional of the 1–level system can be used for the 2–level system. Hence, analogously to (A.1.1), the Laplace functional of the occupation time of the 2–level system is given by

$$E\exp\left\{-\left\langle\int_0^t X_s ds,\varphi\right\rangle\right\} = \exp\{-\langle\!\langle R^1_\infty, \mathbf{u}_\varphi(t)\rangle\!\rangle\}, \quad \varphi\in\mathcal{C}^+_\tau(S), \tag{A.1.15}$$

where $\mathbf{u}_\varphi$ with values in $[0,1]$ is the unique solution of the non-linear evolution equation

$$\mathbf{u}_\varphi(t) = -\frac{V_2}{1+\beta_2}\int_0^t U_{t-s}(\mathbf{u}_\varphi(s)^{1+\beta_2})ds + \int_0^t U_{t-s}(\langle\cdot,\varphi\rangle(1-\mathbf{u}_\varphi(s)))ds. \tag{A.1.16}$$

If follows from (A.1.4) and (A.1.16) that

$$\mathbf{u}_\varphi(t)(\mu) \le \int_0^t U_s(\langle\cdot,\varphi\rangle)(\mu)ds = \int_0^t \langle\mu, T_s\varphi\rangle ds = \langle\mu, G_t\varphi\rangle, \quad \mu\in\mathcal{M}_\tau(S). \tag{A.1.17}$$

A.2. The Palm formula

Let M be a measure on $\mathcal{M}_\tau(S)$ whose intensity measure

$$\Lambda_M := \int_{\mathcal{M}_\tau(S)} \nu(\cdot)M(d\nu)$$

is locally finite, i.e. $\langle\Lambda_H,\varphi\rangle < \infty$ for all $\varphi\in\mathcal{C}_c(S)$. The Palm measures of M are a family $(M_x)_{x\in S}$ of probability measures on $\mathcal{M}_\tau(S)$ which satisfy

$$\int_{\mathcal{M}_\tau(S)} \langle\nu,\varphi\rangle F(\nu)M(d\nu) = \int_S \varphi(x)\int_{\mathcal{M}_\tau(S)} F(\nu)\, M_x(d\nu)\Lambda_M(dx) \tag{A.2.1}$$

for all measurable $F:\mathcal{M}_\tau(S)\to\mathbb{R}_+$ and $\varphi: S\to\mathbb{R}_+$. If M is supported by $\{\mu\in\mathcal{M}_\tau(S)\,|\,\mu$ is $\{0,1,2,\dots,\infty\}$–valued$\}$, then

$$M_x(\{\mu|\,\mu(x)\ge 1\}) = 1$$

for Λ_M–almost all x, and in this case the reduced Palm measures M_x^{red}, $x\in S$, are defined by

$$M_x^{red} = M_x(\{\mu-\delta_x\in(\cdot)\}). \tag{A.2.2}$$

A.3. Tree representations of the Palm measures of R_t^1 and R_∞^1

The Palm measures of R_t^1 and of the equilibrium canonical measure R_∞^1 described in Subsection A.1 have a representation in terms of a backward tree which we recall here.

Lemma A.3.1. (Gorostiza and Wakolbinger[(21)], Theorem 2.3). Let W^x be a random path of the motion process starting in $x\in S$, let π be a random Poisson configuration on $\mathbb{R}_+$ with intensity V, and for each $y\in S$, $r>0$ and $i=1,2,\dots$, let $X_r^{y,i}$ be a branching particle system arising from one ancestor at site y and developing over time r. Let N_r be an integer–valued random variable with generating function $1-(1+\beta)q(1-s)^\beta$ (see Subsection 2.5). Assume all these objects are independent. Then the particle systems

$$\Phi_x^t := \int_0^t \sum_{i=1}^{Nr} X_r^{W_r^x, i}(\cdot)\, \pi(dr) \tag{A.3.1}$$

and

$$\Phi_x^\infty := \int_0^\infty \sum_{i=1}^{Nr} X_r^{W_r^x, i}(\cdot)\, \pi(dr) \tag{A.3.2}$$

have distributions $(R_t^1)_x^{red}$ and $(R_\infty^1)_x^{red}$, respectively, for ρ–almost all $x \in S$.

It follows immediately from (A.3.1), (A.3.2), (A.1.11), (A.1.14) and the Palm formula (A.2.1) that all the moments of R_t^1 increase to those of R_∞^1 as $t \to \infty$.

A.4. Second and third moments of R_∞^1

Proof of (A.1.12) and (A.1.13):

The proof of can be carried out directly by using the explicit form of the Laplace transform of R_∞^1 (as given, e.g., in Gorostiza and Wakolbinger(21), Theorem 3.3). Here we give an argument which uses the structure of R_t^1 in (A.1.9) and the monotone convergence of the moments of R_t^1 mentioned above.

Let us introduce the notation

$$A_{t,x}(\varphi, \psi) = \int_0^t T_s(T_{t-s}\varphi \cdot T_{t-s}\psi)(x) ds, \tag{A.4.1}$$

$$B_{t,x}(\varphi, \psi, \zeta) = \int_0^t T_s \left(T_{t-s}\varphi \left(\int_0^{t-s} T_{t-s-r}\left(T_r\psi \cdot T_r\zeta\right) dr \right)\right)(x) ds. \tag{A.4.2}$$

The following formulae, which can be obtained either by differentiating the Laplace functional of X_t^x or by means of a tree representation of the Palm measures of the distribution of X_t^x (similar to (A.3.1)), are well known (e.g. Klenke(30), Lemma 3.1),

$$E[\langle X_t^x, \varphi\rangle \langle X_t^x, \psi\rangle] = T_t(\varphi\psi)(x) + V A_{t,x}(\varphi, \psi), \tag{A.4.3}$$

$$\begin{aligned} &E[\langle X_t^x, \varphi\rangle \langle X_t^x, \psi\rangle \langle X_t^x, \zeta\rangle] \\ &= T_t(\varphi\psi\zeta)(x) + V(A_{t,x}(\varphi, \psi\zeta) + A_{t,x}(\psi, \varphi\zeta) + A_{t,x}(\zeta, \varphi\psi)) \\ &+ V^2(B_{t,x}(\varphi, \psi, \zeta) + B_{t,x}(\psi, \varphi, \zeta) + B_{t,x}(\zeta, \varphi, \psi)). \end{aligned} \tag{A.4.4}$$

Note that (A.4.3) is just a rewriting of (A.1.7). We need the following lemma.

Lemma A.4.1.

(a) $\displaystyle\int_S A_{t,x}(\varphi,\psi)\rho(dx) \longrightarrow \frac{1}{2}\langle \rho, \varphi G\psi\rangle$ as $t\to\infty$.

(b) $\displaystyle\int_S B_{t,x}(\varphi,\psi,\zeta)\rho(dx) \longrightarrow \frac{1}{2}\left\langle \rho, \varphi \int_0^\infty GT_r(T_r\psi\cdot T_r\zeta)dr\right\rangle$ as $t\to\infty$.

Proof:

(a)
$$\int_S A_{t,x}(\varphi,\psi)\rho(dx) = \left\langle \rho, \int_0^t (T_s\varphi\cdot T_s\psi)ds\right\rangle = \left\langle \rho, \varphi\int_0^t T_{2s}(\psi)ds\right\rangle \longrightarrow \frac{1}{2}\langle \rho,\varphi G\psi\rangle \text{ as } t\to\infty.$$

(b)
$$\begin{aligned}
\int_S B_{t,x}(\varphi,\psi,\zeta)\rho(dx) &= \left\langle \rho, \int_0^t T_s\varphi\left(\int_0^s T_{s-r}(T_r\psi\cdot T_r\zeta)dr\right)ds\right\rangle\\
&= \left\langle \rho, \varphi\int_0^t\int_0^s T_{2s-r}(T_r\psi\cdot T_r\zeta)drds\right\rangle\\
&= \left\langle \rho, \varphi\int_0^t\int_r^t T_{2s-r}(T_r\psi\cdot T_r\zeta)dsdr\right\rangle\\
&= \frac{1}{2}\left\langle \rho, \varphi\int_0^t\int_r^{2t-r} T_u(T_r\psi\cdot T_r\zeta)dudr\right\rangle\\
&= \frac{1}{2}\left\langle \rho, \varphi\int_0^t\int_0^{2(t-r)} T_vT_r(T_r\psi\cdot T_r\zeta)dvdr\right\rangle\\
&\longrightarrow \frac{1}{2}\left\langle \rho, \varphi\int_0^\infty GT_r(T_r\psi\cdot T_r\zeta)dr\right\rangle \quad \text{as} \quad t\to\infty. \qquad \square
\end{aligned}$$

Combining Lemma A.4.1 with (A.1.9), (A.4.3), (A.4.4), and using the above mentioned monotonicity of the moments of R_t^1, the proof of (A.1.12) and (A.1.13) is complete. □

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